Kozma–Nitzan Connection Inequalities for Bernoulli Percolation

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Last Updated: 2 September 2026

The mathematical arguments, exposition, and accompanying Lean formalization were produced entirely by ChatGPT 5.6 Sol and Claude Fable 5.1, prompted by Ahmed Bou-Rabee.

Warning: This manuscript has not yet undergone extensive independent proofreading.

Abstract

Kozma and Nitzan reduced the question whether critical Bernoulli percolation has an infinite open cluster to connection inequalities on finite weighted graphs. We prove all five of their conjectures and answer Questions 5, 7, and 9 affirmatively. Question 8 has a negative answer under the reading that the requested inequality must hold for every minimizing vertex: a four-vertex graph gives a counterexample. The main argument is a comparison principle for increasing functions of an open cluster, valid even when the cluster is conditioned to avoid prescribed vertices. We also prove a version for the union of the clusters of finitely many starting vertices. These two forms yield the strong pre-FKG inequality, allow the minimizer in Conjecture 4 to be chosen before conditioning, remove the two auxiliary inequalities in display (39) from Conjecture 6, give one coefficient vector that works simultaneously for every target in Question 5, and prove Question 9 for every minimizer.

1 Introduction

In Bernoulli percolation on a finite weighted graph, each edge is open independently with its prescribed probability. Write $x \leftrightarrow y$ when vertices x and y are joined by an open path, and write C_x for the open cluster of x. Fix two vertices o and b, and think of a nonempty set A as a collection of possible intermediate vertices between o and b. The simplest inequality considered here is

$$\mathbb{P}(o \leftrightarrow b) \geq \mathbb{P}(o \leftrightarrow A) \min_{x \in A} \mathbb{P}(x \leftrightarrow b)\,.$$

The tempting picture is to follow an open path from o until it first reaches A and then continue toward b. The difficulty is that the vertex of A selected by the first path is chosen by the same random configuration that must supply the second path. Thus there are not two independent connection probabilities to multiply. This dependence created by selecting a vertex of A through C_o is the probabilistic obstacle in Kozma and Nitzan's reduction of the critical-percolation problem [KN24].

Kozma and Nitzan separated the problem into five conjectures and four questions. We prove Conjectures 1, 2, 3, 4, and 6, and we answer Questions 5, 7, and 9 affirmatively. Several conclusions are stronger than requested. The pre-FKG inequality retains the event $\{o \leftrightarrow A\}$ on its left side; the minimizing vertex in Conjecture 4 can be selected before that event is imposed; the hypotheses in display (39) of Conjecture 6 are automatic; and the coefficients in Question 5 satisfy a sharper inequality in which o is connected to both A and b. Question 8 is the sole exception. Under the reading that every minimizer must work, a four-vertex graph disproves it as stated.

We begin with a concrete comparison. Let F be an increasing function of a cluster, and order the vertices of a finite set T by the values $\mathbb{E}[F(C_x)]$. For each $x \in T$, let J_x be the event that x is the first vertex in this ordering that belongs to C_o. We prove

$$\sum_{x \in T} \mathbb{P}(J_x)\, \mathbb{E}[F(C_x)] \leq \mathbb{E}\big[F(C_o);\, o \leftrightarrow T\big]\,.$$

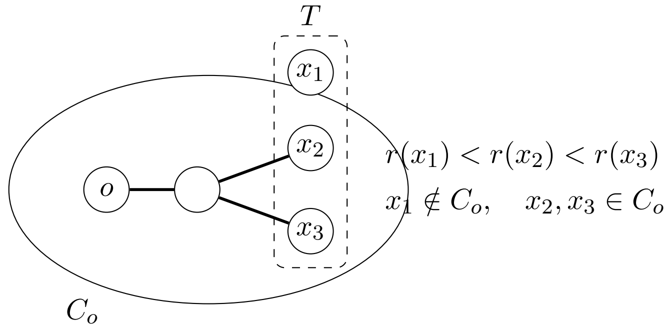

Figure 1: The rank r is fixed before the configuration is sampled. Here $r(x_1) < r(x_2) < r(x_3)$ and $T \cap C_o = \{x_2, x_3\}$, so x_2 is the smallest-ranked vertex of $T \cap C_o$ and J_{x_2} occurs. The schematic layout carries no metric meaning: this is not a first-hit statement along a path.

Figure 1 illustrates this fixed-order selection rule. Although $C_o = C_x$ on J_x, conditioning on J_x changes the law of the cluster; the inequality controls this dependence. We first prove a form conditioned on the cluster avoiding a fixed set Y. The increasing function Φ_a introduced in Section 5 then converts conditional expectations into differences of unconditional expectations. The result is the following useful principle: if $a \in A$ minimizes $\mathbb{E}[F(C_x)]$ over $x \in A$, then

$$\mathbb{E}\big[F(C_a);\, o \leftrightarrow A\big] \le \mathbb{E}\big[F(C_o);\, o \leftrightarrow A\big]\,.$$

Taking F to indicate connection to b proves Conjectures 1–3 and Question 7. General increasing functions give Conjecture 4, while finite-dimensional separation produces the single coefficient vector required in Question 5.

Conjecture 6 and Question 9 require the same comparison with several starting vertices. We replace C_o by $C_R = \bigcup_{x \in R} C_x$ for a nonempty set R. When R consists of the two endpoints of an edge, C_R does not depend on whether that edge is open; forcing the edge open therefore gives Conjecture 6. For Question 9, we expose the edges incident to o. Conditional on their states, C_o is the union of the clusters starting from o and its open neighbors, so the comparison for C_R applies.

The exposition does not assume familiarity with the Kozma–Nitzan paper. Section 2 defines the model and states all nine items in full. Section 3 states the correlation inequalities used as inputs. Sections 4–6 prove the comparison for one starting vertex and its first applications. Sections 7 and 8 treat several starting vertices. Section 9 gives the counterexample to Question 8. Section 10 describes the accompanying Lean verification.

2 Model and Statements

We now state the model and the nine Kozma–Nitzan problems in full, making the paper independent of the notation of [KN24].

Let V be a finite set. Each two-element subset $e \subseteq V$ is an edge. Fix an edge-opening probability $p_e \in [0,1]$ and declare the edges open independently, with e open with probability p_e. An edge with $p_e = 0$ may be regarded as absent. Write $\mathbb{P}$ and $\mathbb{E}$ for the resulting probability and expectation.

For vertices x and y, write $x \leftrightarrow y$ when there is a path of open edges from x to y, and $x \not\leftrightarrow y$ otherwise. Connection is reflexive, so $x \leftrightarrow x$ always. Write $C_x = \{y \in V : x \leftrightarrow y\}$ for the open cluster of x, and $\mathcal{C}_x$ for the set of open edges with at least one endpoint in C_x. For an edge set K, let $\operatorname{span} K$ be the set of endpoints of edges in K, so that $C_x = \{x\} \cup \operatorname{span} \mathcal{C}_x$.

For vertex sets $R, B \subseteq V$, put $C_R = \bigcup_{r \in R} C_r$ and write $\{R \leftrightarrow B\} = \{C_R \cap B \neq \varnothing\}$; write $\{R \not\leftrightarrow B\}$ for its complement. For $R = \{x\}$ we abbreviate these events to $\{x \leftrightarrow B\}$ and $\{x \not\leftrightarrow B\}$. Write $\mathcal{C}_R$ for the set of open edges with at least one endpoint in C_R. The indicator of an event is written $\mathbf{1}_A$ when the event has been given the name A, and $\mathbf{1}\{\,\cdot\,\}$ when it is written out. A function F from subsets of V to the real numbers is increasing if $F(K) \le F(L)$ whenever $K \subseteq L$. Every set in this note is finite because V is finite, and A is always a nonempty subset of V.

Kozma and Nitzan write $\mathbb{P}(0 \leftrightarrow b)$ where we write $\mathbb{P}(o \leftrightarrow b)$. They call a function f of a vertex and a configuration a monotone cluster property if $f(v, \omega)$ depends only on the open cluster of v and is increasing under inclusion of that cluster, and if $f(v, \omega) = f(w, \omega)$ whenever the edge $\{v, w\}$ is open. Their five conjectures and four questions are the following; there is no Conjecture 5 and no Question 6.

Conjecture 1 (Kozma and Nitzan, display (1))**.** Let o and b be vertices and let A be a nonempty set of vertices. Then

$$\mathbb{P}(o \leftrightarrow b) \ge \mathbb{P}(o \leftrightarrow A) \min_{x \in A} \mathbb{P}(x \leftrightarrow b)\,.$$

For each $x \in A$, Harris's inequality gives $\mathbb{P}(o \leftrightarrow A, x \leftrightarrow b) \ge \mathbb{P}(o \leftrightarrow A)\mathbb{P}(x \leftrightarrow b)$. Thus Conjecture 2 is formally stronger than Conjecture 1.

Conjecture 2 (Kozma and Nitzan, display (2))**.** Let o and b be vertices and let A be a nonempty set of vertices. Then

$$\mathbb{P}(o \leftrightarrow b) \ge \min_{x \in A} \mathbb{P}(o \leftrightarrow A,\ x \leftrightarrow b)\,.$$

Kozma and Nitzan call Conjecture 1 the post-FKG conjecture and Conjecture 2 the pre-FKG conjecture. They record the formally stronger form

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \ge \min_{x \in A} \mathbb{P}(o \leftrightarrow A,\ x \leftrightarrow b)\,, \tag{1}$$

their display (3).

Conjecture 3 (Kozma and Nitzan, page 15)**.** For every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every finite weighted graph with vertex set V, every nonempty $A \subseteq V$, and all $o, b \in V$, the following implication holds: if $\mathbb{P}(o \leftrightarrow A) > 1 - \delta$ and $\mathbb{P}(x \leftrightarrow b) > 1 - \delta$ for every $x \in A$, then $\mathbb{P}(o \leftrightarrow b) > 1 - \varepsilon$.

Conjecture 4 (Kozma and Nitzan, page 32)**.** Let f be a monotone cluster property, let A be a nonempty set of vertices and let o be a vertex. Then

$$\mathbb{E}\big[f(o, \cdot);\, o \leftrightarrow A\big] \ge \min_{x \in A} \mathbb{E}\big[f(x, \cdot);\, o \leftrightarrow A\big]\,.$$

For an edge e, write $\mathbb{P}^e$ and $\mathbb{E}^e$ for probability and expectation under the law obtained by setting $p_e = 1$ and leaving all other edge probabilities unchanged. Kozma and Nitzan write $G \downarrow e$ for the corresponding graph; for connection events, this is equivalent to contracting e.

Conjecture 6 (Kozma and Nitzan, displays (39) and (40))**.** Let A be a nonempty set of vertices, let b be a vertex, let $a \in A$ satisfy $\mathbb{P}(a \leftrightarrow b) \le \mathbb{P}(x \leftrightarrow b)$ for every $x \in A$, and let $e = \{v, w\}$ be an edge. Assume

$$\mathbb{P}(x \leftrightarrow b) \ge \mathbb{P}(x \leftrightarrow A)\,\mathbb{P}(a \leftrightarrow b) \qquad \text{for } x = v \text{ and } x = w\,. \tag{2}$$

Then

$$\mathbb{P}^e(v \leftrightarrow b) \ge \mathbb{P}^e(v \leftrightarrow A)\,\mathbb{P}^e(a \leftrightarrow b)\,. \tag{3}$$

Question 5 (Kozma and Nitzan, display (38))**.** Let A be a nonempty set of vertices and let o be a vertex. Do there exist coefficients $(c_x)_{x \in A}$, depending on A, o, and the edge probabilities but not on b, such that $c_x \ge 0$, $\sum_{x \in A} c_x = 1$, and, for every $b \in V$,

$$\mathbb{P}(o \leftrightarrow b) \ge \sum_{x \in A} c_x\, \mathbb{P}(o \leftrightarrow A,\ x \leftrightarrow b)?$$

The last three questions all ask whether a specified vertex $a \in A$ satisfies the inequality of the paper's display (41),

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \geq \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)\,. \tag{4}$$

Question 7 (Kozma and Nitzan, page 36)**.** Let A be a nonempty set of vertices, let o and b be vertices, and let $a \in A$ satisfy $\mathbb{P}(a \leftrightarrow b) \leq \mathbb{P}(x \leftrightarrow b)$ for every $x \in A$. Must

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \geq \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)$$

hold?

Question 8 (Kozma and Nitzan, page 36)**.** Let A be a nonempty set of vertices and let $o, b \in V$. Must the following inequality hold for every $a \in A$ satisfying $\mathbb{P}(a \leftrightarrow b,\ o \not\leftrightarrow A) \leq \mathbb{P}(x \leftrightarrow b,\ o \not\leftrightarrow A)$ for every $x \in A$?

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \geq \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)$$

For a vertex o, let $\mathbb{P}_\star$ and $\mathbb{E}_\star$ be the probability and the expectation in the percolation measure in which every edge incident to o is closed and all other edge probabilities are unchanged.

Question 9 (Kozma and Nitzan, page 36)**.** Let A be a nonempty set of vertices, let o and b be vertices, and let $a \in A$ satisfy $\mathbb{P}_\star(a \leftrightarrow b) \leq \mathbb{P}_\star(x \leftrightarrow b)$ for every $x \in A$. With both probabilities evaluated in the original measure, must

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \geq \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)$$

hold?

The quantity minimized in Question 8 is the probability that x is connected to b and o is not connected to A. It differs from the quantity on the right side of (4), which is treated by (1).

Section 3 states the correlation inequalities that control the dependence between the cluster reaching A and the choice of a vertex in A.

3 Correlation Inputs

We use two families of correlation inequalities. We state them in the precise forms used in Sections 4 and 7, then record the conditional identities needed when several clusters are explored at once.

The first input is the conditional positive-association inequality of van den Berg, Häggström and Kahn [BHK06].

Theorem 3.1 (van den Berg, Häggström and Kahn)**.** *Let s be a vertex, let X_1 and X_2 be sets of vertices not containing s and let g_1 and g_2 be nonnegative increasing functions of edge sets. Then*

$$\mathbb{E}\big[g_1(\mathcal{C}_s);\ s \not\leftrightarrow X_1\big]\,\mathbb{E}\big[g_2(\mathcal{C}_s);\ s \not\leftrightarrow X_2\big] \leq \mathbb{E}\big[g_1(\mathcal{C}_s)g_2(\mathcal{C}_s);\ s \not\leftrightarrow (X_1 \cap X_2)\big]\,\mathbb{P}\big(s \not\leftrightarrow (X_1 \cup X_2)\big)\,.$$

A companion inequality of the same authors governs two clusters at once.

Theorem 3.2 (van den Berg, Häggström and Kahn)**.** *Let S_1 and S_2 be sets of vertices, and let g_3 and g_4 be real functions of two edge sets, both increasing in the first argument and decreasing in the second. Let $D = \{S_1 \not\leftrightarrow S_2\}$ and, for $i = 3, 4$, set $G_i(\omega) = g_i(\mathcal{C}_{S_1}(\omega), \mathcal{C}_{S_2}(\omega))$. Then*

$$\mathbb{E}[G_3; D]\,\mathbb{E}[G_4; D] \leq \mathbb{P}(D)\,\mathbb{E}[G_3 G_4; D]\,.$$

Exchanging S_1 and S_2 shows that Theorem 3.2 also holds when g_3 and g_4 are both decreasing in the first argument and increasing in the second. Replacing g_4 by $-g_4$ reverses the inequality:

$$\mathbb{P}(S_1 \not\leftrightarrow S_2)\,\mathbb{E}\big[g_3 g_4;\ S_1 \not\leftrightarrow S_2\big] \leq \mathbb{E}\big[g_3;\ S_1 \not\leftrightarrow S_2\big]\,\mathbb{E}\big[g_4;\ S_1 \not\leftrightarrow S_2\big]\,, \tag{5}$$

when one of the two functions is increasing in the first argument and decreasing in the second while the other is decreasing in the first and increasing in the second.

Taking $X_1 = X_2$ in Theorem 3.1 gives the conditional positive association of two increasing functions of a single cluster,

$$\mathbb{E}\big[g_1(\mathcal{C}_s);\, s \nleftrightarrow X\big]\,\mathbb{E}\big[g_2(\mathcal{C}_s);\, s \nleftrightarrow X\big] \le \mathbb{P}(s \nleftrightarrow X)\,\mathbb{E}\big[g_1(\mathcal{C}_s)g_2(\mathcal{C}_s);\, s \nleftrightarrow X\big]\,. \tag{6}$$

Because the edge-cluster state space is finite, we may add constants that make g_1 and g_2 nonnegative. Adding a constant to either function changes the two sides of (6) by the same quantity. Thus (6) holds for arbitrary real-valued increasing g_1, g_2. Taking $g_1 = g_2 = 1$ in Theorem 3.1 gives Harris's inequality for the two avoidance events $\{s \nleftrightarrow X_1\}$ and $\{s \nleftrightarrow X_2\}$. We also use Harris's inequality [H60] for increasing events and the Ahlswede–Daykin four-functions theorem [AD78].

The second input is a conditional covariance inequality with finitely many correction terms. Theorem 3.3 is imported from the machine-checked proof [Dev26]. We state its precise probabilistic form so that no normalization or conditioning convention is left implicit; Section 10 gives the corresponding Lean declaration.

Fix a vertex s, a set Y of vertices with $s \notin Y$, a list $z_1, \dots, z_\ell$ of distinct vertices and two further distinct vertices o and v, all the named vertices lying outside Y and being pairwise distinct. Write ν for the conditional law $\mathbb{P}(\,\cdot \mid s \nleftrightarrow Y)$ and put

$$\pi_j(u) = \mathbb{P}\big(u \in C_{z_j} \bigm| z_j \nleftrightarrow B_j\big), \qquad \tau = \mathbb{P}\big(o \in C_v \bigm| v \nleftrightarrow B_{\ell+1}\big)\,, \tag{7}$$

where $B_1 = \{s\} \cup Y$ and $B_{j+1} = B_j \cup \{z_j\}$ for $1 \le j \le \ell$. For a real function q on V put $q^0 = q$ and

$$q^j(u) = q^{j-1}(u) - \pi_j(u)\, q^{j-1}(z_j) \quad \text{for } 1 \le j \le \ell, \qquad \mathfrak{M}[q] = q^\ell(o) - \tau\, q^\ell(v)\,. \tag{8}$$

The jth step subtracts from the value at u the value at z_j, multiplied by the conditional connection probability $\pi_j(u)$. All constants are computed under $\mathbb{P}$, not under ν.

Theorem 3.3. *Let V be a finite vertex set with every edge probability strictly between zero and one, let $Y \subseteq V$, let s be a vertex outside Y, let $z_1, \dots, z_\ell$ be distinct vertices outside $Y \cup \{s\}$, let o and v be two further distinct vertices outside $Y \cup \{s, z_1, \dots, z_\ell\}$, and let g be an increasing function of edge sets. Put $B_1 = \{s\} \cup Y$ and $B_{j+1} = B_j \cup \{z_j\}$. Define*

$$\pi_j(u) = \mathbb{P}\big(u \in C_{z_j} \mid z_j \nleftrightarrow B_j\big), \qquad \tau = \mathbb{P}\big(o \in C_v \mid v \nleftrightarrow B_{\ell+1}\big),$$

and let $\nu = \mathbb{P}(\,\cdot \mid s \nleftrightarrow Y)$. Set $\kappa(u) = \mathrm{Cov}_\nu(g(\mathcal{C}_s), \mathbf{1}\{u \in C_s\})$, and recursively define $\kappa^0 = \kappa$ and

$$\kappa^j(u) = \kappa^{j-1}(u) - \pi_j(u)\kappa^{j-1}(z_j) \qquad (1 \le j \le \ell).$$

Then

$$\kappa^\ell(o) - \tau\kappa^\ell(v) \ge 0.$$

For $\ell = 0$ the conclusion reads

$$\mathrm{Cov}_\nu\big(g(\mathcal{C}_s), \mathbf{1}\{o \in C_s\}\big) \ge \mathbb{P}\big(o \in C_v \bigm| v \nleftrightarrow \{s\} \cup Y\big)\ \mathrm{Cov}_\nu\big(g(\mathcal{C}_s), \mathbf{1}\{v \in C_s\}\big)\,,$$

If moreover $Y = \varnothing$ and $g(K) = \mathbf{1}\{v \in \{s\} \cup \operatorname{span} K\}$, the displayed inequality is equivalent to Harris's inequality for the increasing events $\{s \leftrightarrow v\}$ and $\{s \leftrightarrow o\} \cup \{v \leftrightarrow o\}$.

For the remaining two inputs, fix a vertex s, a set Y not containing s, and an increasing function g of edge sets. We first use a cluster-exploration consequence of the domain Markov property, in the form of BHK's cluster-conditioning lemma [BHK06, Lemma 2.4]. Condition on $\mathcal{C}_Y$. On the event $D = \{s \nleftrightarrow Y\}$, the vertex set $C_Y = Y \cup \operatorname{span} \mathcal{C}_Y$ is disjoint from C_s, every edge with exactly one endpoint in C_Y is closed, and the edges with both endpoints outside C_Y retain their original probabilities and are independent of the edges meeting C_Y.

Write $\bar{g}$ for the expectation of $g(\mathcal{C}_s)$ under the product measure in which every edge meeting C_Y is closed and every remaining edge keeps its original probability. Then the random variable

$$\Theta = \mathbf{1}_D\big(g(\mathcal{C}_s) - \bar{g}\big) \tag{9}$$

has $\mathbb{E}[\Theta \mid \mathcal{C}_Y] = 0$. For a set K of vertices disjoint from $\{s\} \cup Y$, let $\Psi(K)$ be minus the expectation of Θ under the product measure in which every edge meeting K is closed. Then $\Psi(\varnothing) = 0$, the function Ψ is nonnegative and increasing, and for every vertex z and every set B containing $\{s\} \cup Y$ but not z,

$$\mathbb{E}\big[\Theta;\, z \not\leftrightarrow B\big] = -\,\mathbb{E}\big[\Psi(C_z);\, z \not\leftrightarrow B\big]\,. \tag{10}$$

The conditional-mean identity and (10) express the domain Markov property: given $\mathcal{C}_B$, the edges with both endpoints outside C_B retain their original probabilities and are independent of the exposed cluster.

We also need the following stopped conditional covariance estimate. It follows from Gladkov's decision-tree Harris–Kleitman inequality [Gla24, Theorem 3.2] together with the BHK cluster inequalities; its machine-checked formulation is identified in Section 10. For a set W of vertices write $\mathbb{P}_W$, $\mathbb{E}_W$ and Cov_W for the probability, the expectation and the covariance in the percolation measure in which every edge with an endpoint outside W is closed and every edge with both endpoints in W keeps its probability. For a set X of vertices containing s and a vertex $v \notin X$, put

$$\eta_v(W) = \mathrm{Cov}_W\big(g(\mathcal{C}_s), \mathbf{1}\{v \leftrightarrow X\}\big) \ \text{ for } v \in W, \qquad \eta_v(W) = 0 \ \text{ for } v \notin W\,. \tag{11}$$

Then for every W and every $Q \subseteq W$,

$$\mathbb{E}_W\big[\mathbf{1}\{s \not\leftrightarrow Q\}\, \eta_v(W \setminus C_Q)\big]\, \mathbb{P}_W(v \not\leftrightarrow X) \le \mathbb{P}_W\big(v \not\leftrightarrow X \cup Q\big)\, \eta_v(W)\,, \tag{12}$$

and the left side of (12) decreases as Q grows.

Theorem 4.1 turns these correlation inputs into a comparison for the vertex of T selected by the cluster of o. The conditioning on avoiding Y is essential: in Section 5 we take $Y = \{a\}$, where a is the proposed minimizer.

4 A Conditional Comparison

Our first main estimate controls the bias caused by selecting a vertex of T through the cluster of o. We prove it while requiring that cluster to miss a fixed set Y.

Throughout this section V is a finite vertex set with edge probabilities in $[0, 1]$, the set Y is a subset of V, and F is an increasing function of subsets of V. For a vertex x with $\mathbb{P}(x \not\leftrightarrow Y) > 0$ put

$$m_x = \frac{\mathbb{E}\big[F(C_x);\, x \not\leftrightarrow Y\big]}{\mathbb{P}(x \not\leftrightarrow Y)}\,, \tag{13}$$

the conditional expectation of $F(C_x)$ given $x \not\leftrightarrow Y$. Also fix a set $T \subseteq V \setminus Y$ with $\mathbb{P}(x \not\leftrightarrow Y) > 0$ for every $x \in T$, and an injective real function r on T with $m_x \le m_y$ whenever $r(x) < r(y)$. Such an r exists: order T by the values m_x and break ties arbitrarily. For $x \in T$ and a vertex o write

$$J_x = \{o \not\leftrightarrow Y\} \cap \{o \leftrightarrow x\} \cap \bigcap_{y \in T,\ r(y) < r(x)} \{o \not\leftrightarrow y\} \tag{14}$$

for the event that o is not connected to Y and x is the vertex of smallest value of r in $T \cap C_o$. The events J_x are disjoint and their union is $\{o \not\leftrightarrow Y\} \cap \{o \leftrightarrow T\}$. Put

$$\Lambda_o(T) = \mathbb{E}\big[F(C_o);\, o \not\leftrightarrow Y,\ o \leftrightarrow T\big] - \sum_{x \in T} \mathbb{P}(J_x)\, m_x\,. \tag{15}$$

On J_x the vertex x has the smallest value of r among the vertices of T in C_o, hence the smallest value of m, and on J_x the clusters C_o and C_x coincide. Consequently (15) admits the expression

$$\Lambda_o(T) = \mathbb{E}\Big[\mathbf{1}\{o \not\leftrightarrow Y,\, o \leftrightarrow T\}\Big(F(C_o) - \min_{x \in T \cap C_o} m_x\Big)\Big]\,, \tag{16}$$

which does not refer to r. If $T \cap C_o = \varnothing$, the indicator in (16) is zero, so assigning the value zero to the empty minimum does not affect the expectation.

Two cases are immediate. If $o \in Y$, then the event $\{o \not\leftrightarrow Y\}$ is empty and $\Lambda_o(T) = 0$. If $o \in T$, then $o \in T \cap C_o$ on $\{o \not\leftrightarrow Y\}$, so the minimum in (16) is at most m_o, whence $\Lambda_o(T) \ge \mathbb{E}[F(C_o) - m_o;\, o \not\leftrightarrow Y] = 0$ by (13).

Theorem 4.1. *Let V be the vertex set of a finite weighted graph, let $Y \subseteq V$, let $F : 2^V \to \mathbb{R}$ be increasing, and let $T \subseteq V \setminus Y$ satisfy $\mathbb{P}(x \nleftrightarrow Y) > 0$ for every $x \in T$. Set*

$$m_x = \frac{\mathbb{E}[F(C_x); x \nleftrightarrow Y]}{\mathbb{P}(x \nleftrightarrow Y)}$$

and choose an injective $r : T \to \mathbb{R}$ such that $m_x \le m_y$ whenever $r(x) < r(y)$. For $o \in V$ and $x \in T$, set

$$J_x = \{o \nleftrightarrow Y, o \leftrightarrow x\} \cap \bigcap_{\substack{y \in T \\ r(y) < r(x)}} \{o \nleftrightarrow y\}.$$

Then

$$\sum_{x \in T} \mathbb{P}(J_x)\, m_x \le \mathbb{E}\big[F(C_o);\ o \nleftrightarrow Y,\ o \leftrightarrow T\big]\,.$$

For $Y = \varnothing$ this is the inequality quoted in the introduction. No sign hypothesis on F is needed. The proof removes from T the vertex with the largest value of m. The following consequence of Theorem 3.3 controls the possible negative term introduced by this removal.

Proposition 4.2. *Let V be the vertex set of a finite weighted graph, let $Y \subseteq V$, let $F : 2^V \to \mathbb{R}$ be increasing, and let $T \subseteq V \setminus Y$ satisfy $\mathbb{P}(x \nleftrightarrow Y) > 0$ for every $x \in T$. Define*

$$m_x = \frac{\mathbb{E}[F(C_x); x \nleftrightarrow Y]}{\mathbb{P}(x \nleftrightarrow Y)}$$

and let $r : T \to \mathbb{R}$ be injective with $m_x \le m_y$ whenever $r(x) < r(y)$. For $u \in V$, define

$$\Lambda_u(T) = \mathbb{E}[F(C_u); u \nleftrightarrow Y, u \leftrightarrow T] - \sum_{x \in T} \mathbb{P}\left(\{u \nleftrightarrow Y, u \leftrightarrow x\} \cap \bigcap_{\substack{y \in T \\ r(y) < r(x)}} \{u \nleftrightarrow y\}\right) m_x.$$

If $v \in V \setminus (Y \cup T)$, then, for every $o \in V$,

$$\mathbb{P}\big(v \nleftrightarrow Y \cup T,\ o \leftrightarrow v\big)\, \Lambda_v(T) \le \mathbb{P}\big(v \nleftrightarrow Y \cup T\big)\, \Lambda_o(T)\,.$$

Proof. We proceed in three steps: an induction on the cardinality of T, its specialization to an empty list of auxiliary vertices, and the passage to weights in $[0, 1]$.

Assume first that every edge probability lies strictly between zero and one, that o lies outside $Y \cup T$ and that $o \neq v$. Let $z_1, \dots, z_\ell$ be distinct vertices outside $Y \cup T \cup \{o, v\}$. Let $\mathfrak{M}$ be the functional (8) built from the constants (7) with Y replaced by $Y \cup T$ at every step. We start by showing, by induction on the cardinality of T and simultaneously for every such list and every pair of vertices o and v, that

$$\mathfrak{M}\big[\, u \mapsto \Lambda_u(T) \,\big] \ge 0\,. \tag{17}$$

In the base case $T = \varnothing$ each $\Lambda_u(\varnothing)$ vanishes. For the inductive step, let T be nonempty, let $k \in T$ have the largest value of r, and put $T_0 = T \setminus \{k\}$. Abbreviate

$$P = \mathbb{P}(k \nleftrightarrow Y \cup T_0), \qquad P(u) = \mathbb{P}(k \nleftrightarrow Y \cup T_0,\ u \leftrightarrow k),$$

$$I = \mathbb{E}\big[F(C_k);\ k \nleftrightarrow Y \cup T_0\big], \qquad I(u) = \mathbb{E}\big[F(C_k);\ k \nleftrightarrow Y \cup T_0,\ u \leftrightarrow k\big]\,.$$

Since k has the largest value of r, adding k to T_0 leaves the events J_x for $x \in T_0$ unchanged and creates the single new event $\{k \nleftrightarrow Y \cup T_0,\ u \leftrightarrow k\}$: on $\{u \leftrightarrow k\}$ the clusters of u and k coincide. On that event $F(C_u) = F(C_k)$, so

$$\Lambda_u(T) = \Lambda_u(T_0) + I(u) - P(u)\, m_k\,. \tag{18}$$

Write $\mathrm{Cov}^*(u)$ for $P\, I(u) - I\, P(u)$. This is the unnormalized conditional covariance in Theorem 3.3 without denominators, for the vertex k, the set $Y \cup T_0$ and the increasing function $\mathcal{C} \mapsto F(\{k\} \cup \operatorname{span} \mathcal{C})$. Put $\gamma_k = m_k P - I$. Expanding both sides gives

$$P\big(I(u) - P(u)\, m_k\big) = \mathrm{Cov}^*(u) - \gamma_k\, P(u)\,. \tag{19}$$

Apply Theorem 3.3 with starting vertex k, avoided set $Y \cup T_0$, auxiliary list $z_1, \ldots, z_\ell$, and comparison vertices o, v; this gives $\mathfrak{M}[\mathrm{Cov}^*] \geq 0$. To prove $\mathfrak{M}[P] \geq 0$, set $h(K) = \mathbf{1}\{K \neq \varnothing\}$ and

$$q_0 = \mathbb{P}(k \nleftrightarrow Y \cup T_0, \mathcal{C}_k = \varnothing).$$

For every vertex u at which $\mathfrak{M}$ evaluates its argument, $u \neq k$. Hence the restricted expectations corresponding to h are $I_h(u) = P(u)$ and $I_h = P - q_0$, so the unnormalized conditional covariance is $q_0 P(u)$. A second application of Theorem 3.3 gives $q_0 \mathfrak{M}[P] \geq 0$. Closing every edge incident to k has positive probability, so $q_0 > 0$ and $\mathfrak{M}[P] \geq 0$.

Finally,

$$\gamma_k \leq \Lambda_k(T_0)\,. \tag{20}$$

Indeed (13) gives $\mathbb{E}[(F(C_k) - m_k); k \nleftrightarrow Y] = 0$. Splitting $\{k \nleftrightarrow Y\}$ into $\{k \nleftrightarrow Y \cup T_0\}$ and $\{k \nleftrightarrow Y\} \cap \{k \leftrightarrow T_0\}$ turns this into $\gamma_k = \mathbb{E}[(F(C_k) - m_k); k \nleftrightarrow Y, k \leftrightarrow T_0]$. Expanding this restricted expectation over the disjoint events J_x of (14) taken with k in place of o gives $\gamma_k = \Lambda_k(T_0) + \sum_{x \in T_0} \mathbb{P}(J_x)(m_x - m_k)$, in which every summand is at most zero because $r(x) < r(k)$.

Now apply $\mathfrak{M}$ to (18), multiply by $P > 0$ and use (19):

$$P\,\mathfrak{M}\big[\Lambda_\cdot(T)\big] = P\,\mathfrak{M}\big[\Lambda_\cdot(T_0)\big] + \mathfrak{M}[\mathrm{Cov}^*] - \gamma_k\,\mathfrak{M}[P]\,.$$

The middle term is nonnegative. Replace γ_k by the larger $\Lambda_k(T_0)$, which is legitimate because $\mathfrak{M}[P] \geq 0$, and write $P(u) = P\,\pi_k(u)$ with $\pi_k(u) = \mathbb{P}(u \in C_k \mid k \nleftrightarrow Y \cup T_0)$. The right side is then at least P times the value of $\mathfrak{M}$ at $u \mapsto \Lambda_u(T_0)$ for the list obtained by inserting k in front of $z_1, \ldots, z_\ell$, by the recursion in (8) and linearity. That value is nonnegative by the induction hypothesis. Dividing by P gives (17).

We now take $\ell = 0$ in (17). The conclusion is $\Lambda_o(T) \geq \tau \Lambda_v(T)$ with $\tau = \mathbb{P}(o \in C_v \mid v \nleftrightarrow Y \cup T)$. Multiplication by $\mathbb{P}(v \nleftrightarrow Y \cup T) > 0$ gives the inequality in Proposition 4.2. If $o = v$, then the two sides are equal. If o lies in Y or in T, then the event $\{v \nleftrightarrow Y \cup T\} \cap \{o \leftrightarrow v\}$ is empty, so the left side vanishes, while the right side is nonnegative by the two cases settled after (16).

It remains to pass to weights in $[0, 1]$. The right side of (16) is a finite sum over configurations. Each summand is the configuration probability, a polynomial in the edge probabilities, multiplied by a value depending on the probabilities only through the finitely many numbers m_x. Each m_x is a ratio of polynomials whose denominator is positive at the given vector of edge probabilities. Both sides of Proposition 4.2 are therefore continuous there. Approximate this vector by vectors with all coordinates strictly inside the unit interval and, for each approximation, choose an injective r ordered by its corresponding values m_x. Formula (16) shows that $\Lambda_u(T)$, and hence both sides of Proposition 4.2, are independent of the tie-breaking order r. The interior case applies to every approximation, and continuity gives the desired inequality at the original vector. ☐

Proof of Theorem 4.1. We prove by induction on the cardinality of T that $\Lambda_o(T) \geq 0$ for every vertex o. In the base case $T = \varnothing$ the quantity (15) vanishes. For the inductive step, let T be nonempty, let $k \in T$ have the largest value of r, put $T_0 = T \setminus \{k\}$, and keep the abbreviations P, $P(u)$, I and $I(u)$ of the proof of Proposition 4.2. The identity (18) with $u = o$ reads $\Lambda_o(T) = \Lambda_o(T_0) + I(o) - m_k P(o)$.

Apply (6) with $s = k$, with $X = Y \cup T_0$ and with the two increasing functions $\mathcal{C} \mapsto F(\{k\} \cup \operatorname{span} \mathcal{C})$ and the indicator of the event that o lies in the cluster of k. The result is $P(o)\,I \leq P\,I(o)$, which rearranges to

$$P\big(m_k P(o) - I(o)\big) \leq P(o)\big(m_k P - I\big) = P(o)\,\gamma_k\,.$$

By (20) and $P(o) \geq 0$ the right side is at most $P(o)\Lambda_k(T_0)$, and by Proposition 4.2 with $v = k$ that is at most $P\,\Lambda_o(T_0)$. Hence $P\,\Lambda_o(T) \geq 0$. If $P > 0$, then $\Lambda_o(T) \geq 0$. If $P = 0$, then $P(o) = 0$ and $I(o) = 0$, so $\Lambda_o(T) = \Lambda_o(T_0)$, which is nonnegative by the induction hypothesis. ☐

We have proved the comparison for conditional expectations. We next convert it into a statement about the unconditional expectations that occur in the conjectures.

5 A Fixed Minimizer

Theorem 4.1 orders vertices using expectations conditioned on avoiding a fixed set. The conjectures instead choose a vertex from unconditional expectations and only afterward impose the event $\{o \leftrightarrow A\}$. We now bridge that difference and then deduce Conjectures 1–4 and Question 7.

The obstacle is that the numbers m_x of (13) are conditional expectations, whereas Conjecture 4 compares unconditional ones. The following function removes the difference. For a vertex set K, let $\mathbb{E}^0_K$ denote expectation under the product measure obtained by closing every edge incident to K and retaining the original probability of every other edge. For a vertex a and an increasing F, define

$$\Phi_a(K) = F(K) - \mathbb{E}^0_K[F(C_a)].$$

Lemma 5.1. *Let a and o be vertices and let F be an increasing function of subsets of V. Then Φ_a is increasing, and:*

(i) *for every collection $\mathcal{S}$ of edge configurations,*

$$\mathbb{E}\big[F(C_o) - F(C_a);\, o \nleftrightarrow a,\ \mathcal{C}_o \in \mathcal{S}\big] = \mathbb{E}\big[\Phi_a(C_o);\, o \nleftrightarrow a,\ \mathcal{C}_o \in \mathcal{S}\big]\,;$$

(ii) $\mathbb{E}\big[\Phi_a(C_o);\, o \nleftrightarrow a\big] = \mathbb{E}\big[F(C_o)\big] - \mathbb{E}\big[F(C_a)\big]$.

Proof. Enlarging K increases $F(K)$ and closes more edges, which decreases $\mathbb{E}^0_K[F(C_a)]$, so Φ_a is increasing.

For (i), condition on $\mathcal{C}_o$. On $\{o \nleftrightarrow a\}$ the vertex a lies outside C_o, every edge with exactly one endpoint in C_o is closed, and the edges with both endpoints outside C_o retain their original probabilities. Hence the conditional expectation of $F(C_a)$ given $\mathcal{C}_o$ is $F(C_o) - \Phi_a(C_o)$. Both events in the statement are determined by $\mathcal{C}_o$, so summing over the values of $\mathcal{C}_o$ in $\mathcal{S}$ gives (i).

For (ii), take $\mathcal{S}$ to consist of all edge configurations. On $\{o \leftrightarrow a\}$ the two clusters coincide, so $F(C_o) - F(C_a)$ vanishes there and the left side of (i) equals $\mathbb{E}[F(C_o) - F(C_a)]$. □

The lemma turns unconditional minimality of a into nonnegative conditional expectations. We can therefore apply Theorem 4.1 with $Y = \{a\}$.

Theorem 5.2. *Let A be a nonempty finite set of vertices, let $a \in A$, let o be a vertex, and let F be an increasing function of subsets of V with $\mathbb{E}[F(C_a)] \le \mathbb{E}[F(C_x)]$ for every $x \in A$. Then*

$$\mathbb{E}\big[F(C_a);\, o \leftrightarrow A\big] \le \mathbb{E}\big[F(C_o);\, o \leftrightarrow A\big]\,.$$

Proof. Let T be the set of $x \in A \setminus \{a\}$ with $\mathbb{P}(x \nleftrightarrow a) > 0$. Apply Theorem 4.1 with $Y = \{a\}$, with the set T, with the increasing function Φ_a and with the vertex o. By Lemma 5.1(ii) applied at x in place of o, the number (13) attached to $x \in T$ is $(\mathbb{E}[F(C_x)] - \mathbb{E}[F(C_a)])/\mathbb{P}(x \nleftrightarrow a) \ge 0$. The sum subtracted in (15) is therefore nonnegative, and Theorem 4.1 gives $\mathbb{E}[\Phi_a(C_o);\, o \nleftrightarrow a,\ o \leftrightarrow T] \ge 0$. The event $\{o \leftrightarrow T\}$ is determined by $\mathcal{C}_o$, so Lemma 5.1(i) converts this into

$$\mathbb{E}\big[F(C_o) - F(C_a);\, o \nleftrightarrow a,\ o \leftrightarrow T\big] \ge 0\,. \tag{21}$$

Split $\{o \leftrightarrow A\}$ into $\{o \leftrightarrow a\}$, on which the integrand $F(C_o) - F(C_a)$ vanishes, and $\{o \nleftrightarrow a\} \cap \{o \leftrightarrow A \setminus \{a\}\}$. The event $\{o \nleftrightarrow a, o \leftrightarrow A \setminus \{a\}\}$ differs from $\{o \nleftrightarrow a\} \cap \{o \leftrightarrow T\}$ only by configurations in which o misses a and reaches at least one vertex x of $A \setminus \{a\}$ outside T. Such a configuration lies in the null event $\{x \nleftrightarrow a\}$, and a finite union of null events is null. Hence the left side of (21) equals $\mathbb{E}[F(C_o) - F(C_a); o \leftrightarrow A]$. □

Conjecture 4 follows by choosing a to realize the minimum of $\mathbb{E}[F(C_x)]$ over $x \in A$, provided every monotone cluster property is of the form $\omega \mapsto F(C_v(\omega))$ for an increasing F. Lemma 5.3 establishes this representation. Its hypotheses omit the requirement that $f(v, \omega)$ depend only on C_v, since that follows from the requirement that $f(v, \omega)$ be increasing in C_v.

Lemma 5.3. *Let $f : V \times \Omega \to \mathbb{R}$. Assume that for every $v \in V$ and configurations ω, ξ, the inclusion $C_v(\omega) \subseteq C_v(\xi)$ implies $f(v,\omega) \le f(v,\xi)$. Assume also that for all $v, w \in V$ and every configuration ω, if the edge $\{v,w\}$ is open in ω, then $f(v,\omega) = f(w,\omega)$. Then there is an increasing function $F : 2^V \to \mathbb{R}$ such that $f(v,\omega) = F(C_v(\omega))$ for every v and ω.*

Proof. For a nonempty vertex set S let ω_S be the configuration in which the open edges are exactly those with both endpoints in S. Every vertex of S has open cluster S in ω_S, and for distinct $v, w \in S$ the edge $\{v,w\}$ is open in ω_S. Hence $f(v,\omega_S)$ does not depend on the choice of $v \in S$; call the common value $F(S)$.

If $S \subseteq S_1$ are nonempty and $v \in S$, then $C_v(\omega_S) = S \subseteq S_1 = C_v(\omega_{S_1})$, so $F(S) \le F(S_1)$. Extend F to the empty set by a value at most $F(S)$ for every nonempty S. Finally, for a vertex v and a configuration ω put $S = C_v(\omega)$; then $C_v(\omega_S) = S = C_v(\omega)$, and the hypothesis applied in both directions gives $f(v,\omega) = f(v,\omega_S) = F(S)$. □

We can now translate Theorem 5.2 back into the cluster-property language of Conjecture 4.

Corollary 5.4. *Let V be the vertex set of a finite weighted graph, let f be a monotone cluster property, let $A \subseteq V$ be nonempty, and let $o \in V$. Then*

$$\mathbb{E}[f(o,\cdot); o \leftrightarrow A] \ge \min_{x\in A} \mathbb{E}[f(x,\cdot); o \leftrightarrow A].$$

In particular, this holds for $f(v,\omega) = |C_v|$ and for $f(v,\omega) = \mathbf{1}\{|C_v| \ge n\}$ for every positive integer n.

Proof. By Lemma 5.3 write $f(v,\cdot) = F(C_v)$ with F increasing, choose $a \in A$ realizing the minimum of $\mathbb{E}[F(C_x)]$ over $x \in A$, and apply Theorem 5.2. The two displayed examples are increasing functions of the vertex set. □

The connection inequalities follow by specializing F to an indicator.

Corollary 5.5. *Let V be the vertex set of a finite weighted graph, let $A \subseteq V$ be nonempty, and let $o, b \in V$. Then*

$$\mathbb{P}(o \leftrightarrow b, o \leftrightarrow A) \ge \min_{x\in A} \mathbb{P}(o \leftrightarrow A, x \leftrightarrow b)$$

and

$$\mathbb{P}(o \leftrightarrow b) \ge \mathbb{P}(o \leftrightarrow A) \min_{x\in A} \mathbb{P}(x \leftrightarrow b).$$

If $a \in A$ satisfies $\mathbb{P}(a \leftrightarrow b) \le \mathbb{P}(x \leftrightarrow b)$ for every $x \in A$, then

$$\mathbb{P}(o \leftrightarrow b, o \leftrightarrow A) \ge \mathbb{P}(o \leftrightarrow A, a \leftrightarrow b).$$

Moreover, for every $\varepsilon > 0$, the choice $\delta = \varepsilon/2$ has the following property: if $\mathbb{P}(o \leftrightarrow A) > 1-\delta$ and $\mathbb{P}(x \leftrightarrow b) > 1-\delta$ for every $x \in A$, then $\mathbb{P}(o \leftrightarrow b) > 1-\varepsilon$.

Proof. Take $F = \mathbf{1}\{b \in \cdot\}$, so that $\mathbb{E}[F(C_x); o \leftrightarrow A] = \mathbb{P}(x \leftrightarrow b, o \leftrightarrow A)$ and $\mathbb{E}[F(C_x)] = \mathbb{P}(x \leftrightarrow b)$. Corollary 5.4 is then (1), and dropping the event $\{o \leftrightarrow A\}$ on the left gives Conjecture 2. If $\mathbb{P}(a \leftrightarrow b) \le \mathbb{P}(x \leftrightarrow b)$ for every $x \in A$, then Theorem 5.2 gives $\mathbb{P}(a \leftrightarrow b, o \leftrightarrow A) \le \mathbb{P}(o \leftrightarrow b, o \leftrightarrow A)$, which is (4); so Question 7 has an affirmative answer.

For Conjecture 1, the events $\{x \leftrightarrow b\}$ and $\{o \leftrightarrow A\}$ are both increasing, so the Harris inequality gives $\mathbb{P}(o \leftrightarrow A)\mathbb{P}(x \leftrightarrow b) \le \mathbb{P}(x \leftrightarrow b, o \leftrightarrow A)$ for every $x \in A$. Taking the minimum over $x \in A$ and using (1),

$$\mathbb{P}(o \leftrightarrow A) \min_{x\in A} \mathbb{P}(x \leftrightarrow b) \le \min_{x\in A} \mathbb{P}(x \leftrightarrow b,\ o \leftrightarrow A) \le \mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \le \mathbb{P}(o \leftrightarrow b)\,.$$

For Conjecture 3 take $\delta = \varepsilon/2$. If $\varepsilon \ge 2$, then $1-\varepsilon \le -1 < \mathbb{P}(o \leftrightarrow b)$ and there is nothing to prove. Otherwise $1-\varepsilon/2$ is positive, and Conjecture 1 gives $\mathbb{P}(o \leftrightarrow b) > (1-\varepsilon/2)^2 = 1-\varepsilon+\varepsilon^2/4 \ge 1-\varepsilon$. □

Theorem 5.2 allows the minimizing vertex of A to depend on the target b. Question 5 asks whether a single probability distribution on A works for all targets. Finite-dimensional separation supplies exactly that simultaneous choice.

6 Question 5

Question 5 asks for one probability vector on A that works for every vertex b at once, whereas Corollary 5.4 produces one vertex of A for each b. The passage from one to the other is finite-dimensional convexity. The first step applies Corollary 5.4 not to a single indicator but to a nonnegative weighted count of the vertices of the cluster.

Lemma 6.1. *Let A be a nonempty finite set of vertices, let o be a vertex and let $y_b \geq 0$ for $b \in V$. Then there is a vertex $a \in A$ with*

$$\sum_{b\in V} y_b\, \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b) \leq \sum_{b\in V} y_b\, \mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A)\,.$$

Proof. Put $F(K) = \sum_{b\in V} y_b \mathbf{1}\{b \in K\}$, an increasing function of vertex sets because each y_b is nonnegative. For every vertex x,

$$\mathbb{E}\big[F(C_x);\, o \leftrightarrow A\big] = \sum_{b\in V} y_b\, \mathbb{P}(x \leftrightarrow b,\ o \leftrightarrow A)\,.$$

Choose $a \in A$ minimizing $\mathbb{E}[F(C_x); o \leftrightarrow A]$. The left-hand sum in the lemma is $\mathbb{E}[F(C_a); o \leftrightarrow A]$. Corollary 5.4 bounds it by $\mathbb{E}[F(C_o); o \leftrightarrow A]$, which is the right-hand sum. □

A separation argument turns that family of vertices into a single probability vector.

Theorem 6.2. *Let V be the vertex set of a finite weighted graph. For every nonempty $A \subseteq V$ and $o \in V$, there exist coefficients $(c_x)_{x\in A}$, independent of b, such that $c_x \geq 0$, $\sum_{x\in A} c_x = 1$, and, for every $b \in V$,*

$$\sum_{x\in A} c_x\, \mathbb{P}(o \leftrightarrow A,\ x \leftrightarrow b) \leq \mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \leq \mathbb{P}(o \leftrightarrow b).$$

In particular Question 5 has an affirmative answer.

Proof. The second inequality is event inclusion, so only the first has to be proved. For each $x \in A$ let $M(x, \cdot)$ be the vector indexed by V whose entry at b is $\mathbb{P}(o \leftrightarrow A,\ x \leftrightarrow b)$, and let $\mathbf{w}$ be the vector indexed by V whose entry at b is $\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A)$. Let

$$\mathcal{N} = \operatorname{conv}\{M(x, \cdot) : x \in A\} + \mathbb{R}^V_{\geq 0}.$$

The set $\mathcal{N}$ is convex, and it is closed because it is the sum of a compact set and a closed set in finite dimension.

Suppose $\mathbf{w} \notin \mathcal{N}$. Strict separation of $\mathbf{w}$ from $\mathcal{N}$ gives a vector y indexed by V and a real number t with $\langle y, \mathbf{w}\rangle < t \leq \langle y, \mathbf{z}\rangle$ for every $\mathbf{z} \in \mathcal{N}$. Fix $b \in V$ and fix a point $\mathbf{z}$ of the nonempty set $\mathcal{N}$. Increasing the coordinate b of $\mathbf{z}$ by a positive integer n and leaving the others alone produces another point of $\mathcal{N}$, so $\langle y, \mathbf{z}\rangle + n\, y_b \geq t$ for every n, which forces $y_b \geq 0$.

Lemma 6.1 therefore gives a vertex $a \in A$ with $\langle y, M(a, \cdot)\rangle \leq \langle y, \mathbf{w}\rangle$. But $M(a, \cdot)$ lies in $\mathcal{N}$, so separation gives $\langle y, M(a, \cdot)\rangle \geq t > \langle y, \mathbf{w}\rangle$, a contradiction. Hence $\mathbf{w}$ lies in $\mathcal{N}$: there are $c_x \geq 0$ with $\sum_{x\in A} c_x = 1$ and a vector with nonnegative entries whose sum with $\sum_{x\in A} c_x M(x, \cdot)$ is $\mathbf{w}$. Coordinatewise, this is the first inequality of the theorem. □

Theorems 4.1 and 5.2 begin from one cluster. Conjecture 6 forces an edge open, so its two endpoints must be treated together. This leads to a comparison for the union of the clusters starting from a finite set.

7 Conjecture 6

Recall that $C_R = \bigcup_{s\in R} C_s$ for a finite vertex set R. The union C_R need not be connected.

Theorem 7.1 compares C_R with the cluster of an unconditional minimizer a on configurations where R avoids a and is connected to $A \setminus \{a\}$.

Theorem 7.1. *Let A be a nonempty finite set of vertices, let $a \in A$, let R be a nonempty finite set of vertices, and let F be an increasing function of subsets of V with $\mathbb{E}[F(C_a)] \le \mathbb{E}[F(C_x)]$ for every $x \in A$. Then*

$$\mathbb{E}\big[F(C_R) - F(C_a);\ R \nleftrightarrow a,\ R \leftrightarrow A \setminus \{a\}\big] \ge 0\,. \tag{22}$$

We first deduce Conjecture 6 from the theorem. We then prove the theorem in nine steps. The first four compare a set R with a single vertex after a cluster is exposed. Steps 5–7 extend the conditional covariance inequality from one starting vertex to R. The last two steps repeat the induction over A and remove the conditioning.

7.1 Forcing an Edge

Two invariance facts permit passage to the measure with e forced open. The union of the clusters of the endpoints is unchanged when e is opened, and the expectation of any random variable with this invariance is the same under the two measures. The two cases are shown in Figure 2.

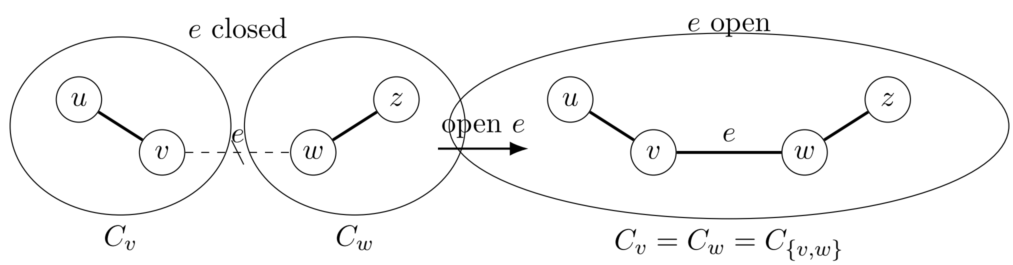

Figure 2: The configurations differ only in the state of $e = \{v, w\}$. In the pictured case, opening e merges C_v and C_w, but in every case it leaves their vertex union $C_{\{v,w\}}$ unchanged. On $\{\{v, w\} \nleftrightarrow a\}$, the cluster C_a is also unchanged. The layout is schematic.

Lemma 7.2. *Let $v, w \in V$ be distinct, let $e = \{v, w\}$, let $a \in V$, and let $B \subseteq V$. For a configuration ω with $e \notin \omega$, write $\omega^e = \omega \cup \{e\}$. Then*

$$C_{\{v,w\}}(\omega^e) = C_{\{v,w\}}(\omega).$$

Consequently, the indicators of $\{\{v, w\} \nleftrightarrow B\}$ and $\{\{v, w\} \leftrightarrow B\}$ agree in ω and ω^e. If $\{v, w\} \nleftrightarrow a$ in ω, then $C_a(\omega^e) = C_a(\omega)$.

Proof. Opening e can only enlarge clusters. Conversely, let a simple path in ω^e join v or w to u. If the path does not use e, it lies in ω. If it uses e, deleting e leaves a final segment in ω from one endpoint of e to u. Thus $u \in C_v(\omega) \cup C_w(\omega)$, proving the equality of cluster unions. The two event identities follow. If $\{v, w\} \nleftrightarrow a$ in ω, no path from a in ω reaches either endpoint of e, so opening e cannot change C_a. □

Lemma 7.3. *Let e be an edge and let φ be a real function of configurations. Assume $\varphi(\omega) = \varphi(\omega \cup \{e\})$ for every configuration ω with $e \notin \omega$. Then $\mathbb{E}^e[\varphi] = \mathbb{E}[\varphi]$.*

Proof. Condition on the states of all edges other than e. They have the same joint law under $\mathbb{P}$ and $\mathbb{P}^e$, since only p_e differs. Given those states, φ has the same value whether e is open or closed. The conditional expectations, and hence the unconditional expectations, agree. $\square$

These two invariance facts let us apply Theorem 7.1 after the edge has been forced open.

Theorem 7.4. *Let A be a nonempty finite set of vertices, let b be a vertex, let $a \in A$ satisfy $\mathbb{P}(a \leftrightarrow b) \le \mathbb{P}(x \leftrightarrow b)$ for every $x \in A$, and let $e = \{v, w\}$ be an edge. Then*

$$\mathbb{P}^e(a \leftrightarrow b,\ v \leftrightarrow A) \le \mathbb{P}^e(v \leftrightarrow b,\ v \leftrightarrow A)\,. \tag{23}$$

Proof. Apply Theorem 7.1 with $R = \{v, w\}$ and $F = \mathbf{1}\{b \in \cdot\}$, for which $\mathbb{E}[F(C_x)] = \mathbb{P}(x \leftrightarrow b)$, so that the hypothesis on a is exactly the minimality assumed here. Define

$$\varphi = (F(C_R) - F(C_a))\mathbf{1}\{R \not\leftrightarrow a, R \leftrightarrow A \setminus \{a\}\}.$$

Lemma 7.2 shows that both event indicators are invariant under opening e. If $R \not\leftrightarrow a$, both C_R and C_a are invariant; otherwise $\varphi = 0$ before and after opening e. Thus φ is independent of the state of e, and Lemma 7.3 identifies its expectations under $\mathbb{P}$ and $\mathbb{P}^e$. Hence

$$\mathbb{E}^e\big[F(C_R) - F(C_a);\ R \not\leftrightarrow a,\ R \leftrightarrow A \setminus \{a\}\big] \ge 0, \qquad R = \{v, w\}\,.$$

Under $\mathbb{P}^e$ the edge e is open with probability one, so almost surely $C_v = C_w = C_R$, the event $\{R \not\leftrightarrow a\}$ is $\{v \not\leftrightarrow a\}$ and the event $\{R \leftrightarrow A \setminus \{a\}\}$ is $\{v \leftrightarrow A \setminus \{a\}\}$. Split $\{v \leftrightarrow A\}$ into $\{v \leftrightarrow a\}$, on which $C_v = C_a$ and the integrand vanishes, and $\{v \not\leftrightarrow a\} \cap \{v \leftrightarrow A \setminus \{a\}\}$. The preceding restricted expectation is therefore $\mathbb{E}^e[\mathbf{1}\{b \in C_v\} - \mathbf{1}\{b \in C_a\}; v \leftrightarrow A] \ge 0$, which is (23). $\square$

Conjecture 6 follows. The events $\{a \leftrightarrow b\}$ and $\{v \leftrightarrow A\}$ are increasing, so the Harris inequality and (23) give

$$\mathbb{P}^e(v \leftrightarrow A)\mathbb{P}^e(a \leftrightarrow b) \le \mathbb{P}^e(a \leftrightarrow b,\ v \leftrightarrow A) \le \mathbb{P}^e(v \leftrightarrow b,\ v \leftrightarrow A) \le \mathbb{P}^e(v \leftrightarrow b)\,,$$

which is (3). Its hypothesis (2) holds automatically: it is the case of Conjecture 1 with x in place of o.

7.2 Starting Vertices

Throughout this subsection fix a vertex s, a set Y of vertices with $s \notin Y$, and an increasing function g of edge sets, and write $D = \{s \not\leftrightarrow Y\}$. Through Step 7, assume that every edge probability lies strictly between zero and one. In particular, every avoidance probability appearing in a denominator below is positive. Step 8 removes this assumption by continuity. For finite sets R, X and B of vertices put

$$H(R; X, B) = \{R \not\leftrightarrow B\} \cap \{R \leftrightarrow X\}\,, \tag{24}$$

the event that C_R is disjoint from B and intersects X. The event $\{R \leftrightarrow s\}$ does not imply $\{R \not\leftrightarrow Y\}$ even when $\{s \not\leftrightarrow Y\}$ holds, because a component of C_R other than the one containing s may meet Y. The two conditions must therefore be carried together. For a single vertex, however, $H(\{u\}; \{s\}, Y) = D \cap \{s \leftrightarrow u\}$, since $u \leftrightarrow s$ makes the clusters of u and s equal. Set

$$\Gamma(R) = \mathbb{P}(D)\,\mathbb{E}\big[g(\mathcal{C}_s);\ H(R; \{s\}, Y)\big] - \mathbb{E}\big[g(\mathcal{C}_s);\ D\big]\,\mathbb{P}\big(H(R; \{s\}, Y)\big)\,, \tag{25}$$

which at $R = \{u\}$ is $\mathbb{P}(D)^2$ times the conditional covariance of Theorem 3.3. The event $H(R; \{s\}, Y)$ is contained in D: on it s lies in C_R and C_R misses Y, so $C_s \subseteq C_R$ misses Y.

The functional (8) is now applied to functions of a set argument. Fix distinct vertices $z_1, \ldots, z_\ell$ and a further vertex v, all outside $\{s\} \cup Y$ and distinct from each other, and recall the sets B_j of (7). Put

$$\pi_j(R) = \frac{\mathbb{P}\big(H(R; \{z_j\}, B_j)\big)}{\mathbb{P}(z_j \not\leftrightarrow B_j)}, \qquad \tau(R) = \frac{\mathbb{P}\big(H(R; \{v\}, B_{\ell+1})\big)}{\mathbb{P}(v \not\leftrightarrow B_{\ell+1})}\,. \tag{26}$$

For a real function q of finite vertex sets put $q^0 = q$, then $q^j(R) = q^{j-1}(R) - \pi_j(R)q^{j-1}(\{z_j\})$ for $1 \le j \le \ell$, and finally, for a prescribed set R_0,

$$\mathfrak{M}[q] = q^\ell(R_0) - \tau(R_0)\, q^\ell(\{v\})\,. \tag{27}$$

At a singleton R_0 the numbers (26) are the constants (7), and $\mathfrak{M}[\Gamma] \ge 0$ is Theorem 3.3. The goal of the first seven steps is the same conclusion for a set R_0 with at least two elements.

Step 1: Linearity. There exist real coefficients $(c_u)_{u\in V}$, depending only on $z_1, \ldots, z_\ell$, v, and R_0, such that

$$\mathfrak{M}[q] = q(R_0) + \sum_{u\in V} c_u q(\{u\})$$

for every q. This follows by induction on ℓ: applying the first subtraction turns $\mathfrak{M}$ into the functional for $z_2, \ldots, z_\ell$ applied to $R \mapsto q(R) - \pi_1(R)q(\{z_1\})$, and every term the subtraction produces is a multiple of $q(\{z_1\})$. Only R_0 enters with a coefficient not attached to a singleton, and that coefficient is one. This is what makes the one-sided estimates below usable.

Step 2: One-Cluster Exploration. For a set U of vertices write $\mathbb{P}_U$ and $\mathbb{E}_U$ as in Section 3. Let X be a set of vertices contained in U and containing s, let $v \in U \setminus X$, and let $R \subseteq U$ be disjoint from $X \cup \{v\}$. For a set K of vertices put

$$\alpha_{R,B}(K) = \mathbf{1}\{K \cap R \neq \varnothing\}\ \mathbb{P}_{U\setminus K}\big(R \setminus K \nleftrightarrow B\big)\,. \tag{28}$$

Then $\alpha_{R,B}(K)$ is increasing in K and decreasing in B, and

$$\mathbb{E}_U\big[\alpha_{R,B}(\mathcal{C}_v);\, v \nleftrightarrow B\big] = \mathbb{P}_U\big(R \nleftrightarrow B,\ R \leftrightarrow v\big)\,. \tag{29}$$

The identity follows by summing over the values of $\mathcal{C}_v$: on $\{v \nleftrightarrow B\}$ the set $K = C_v$ is disjoint from B, the event $\{R \leftrightarrow v\}$ is $\{K \cap R \neq \varnothing\}$, and the edges with both endpoints outside K retain their original probabilities. Applying Theorem 3.1 inside U to the vertex v, to the two sets $X \cup N$ and $X \cup N_1$, and to the functions $\alpha_{R,X\cup N}$ and the constant one, then gives, for disjoint N and N_1,

$$\mathbb{P}_U\big(R \nleftrightarrow X \cup N,\ R \leftrightarrow v\big)\, \mathbb{P}_U\big(v \nleftrightarrow X \cup N_1\big) \le \mathbb{P}_U\big(v \nleftrightarrow X \cup N \cup N_1\big)\, \mathbb{P}_U\big(R \nleftrightarrow X,\ R \leftrightarrow v\big)\,. \tag{30}$$

The two avoided sets meet in X and their union is $X \cup N \cup N_1$.

Step 3: Removing Overlap. For each set $W \subseteq U$, write

$$\varrho(W) = \begin{cases} \dfrac{\mathbb{P}_W(R \nleftrightarrow X,\ R \leftrightarrow v)}{\mathbb{P}_W(v \nleftrightarrow X)}, & \mathbb{P}_W(v \nleftrightarrow X) > 0, \\ 0, & \mathbb{P}_W(v \nleftrightarrow X) = 0. \end{cases}$$

Let $\eta : 2^U \to [0, \infty)$ satisfy $\eta(W) = 0$ whenever $v \notin W$. For sets $Q \subseteq W \subseteq U$, write

$$\phi_W(Q) = \mathbb{E}_W\big[\mathbf{1}\{s \nleftrightarrow Q\}\, \eta(W \setminus C_Q)\big]$$

and

$$\chi_W(Q) = \mathbb{E}_W\big[\mathbf{1}\{s \nleftrightarrow Q\}\mathbf{1}\{R \nleftrightarrow Q\}\, \varrho(W \setminus C_Q)\eta(W \setminus C_Q)\big]\,.$$

All clusters and probabilities in these expressions are computed in the induced model on W. We abbreviate ϕ_U and χ_U to ϕ and χ.

The following lemma removes the disjointness assumption in (30). Its proof is the only place where we use the Ahlswede–Daykin four functions theorem.

Lemma 7.5. *Let U be a finite set of vertices. Let $s, v \in U$, and let $X, R, N, N_1 \subseteq U$ satisfy $s \in X$, $v \notin X$, and $R \cap (X \cup \{v\}) = \varnothing$. Let $\eta : 2^U \to [0, \infty)$ satisfy $\eta(W) = 0$ whenever $v \notin W$. For $Q \subseteq W \subseteq U$, define*

$$\varrho(W) = \begin{cases} \mathbb{P}_W(R \nleftrightarrow X, R \leftrightarrow v)/\mathbb{P}_W(v \nleftrightarrow X), & \mathbb{P}_W(v \nleftrightarrow X) > 0, \\ 0, & \mathbb{P}_W(v \nleftrightarrow X) = 0, \end{cases}$$

$$\phi_W(Q) = \mathbb{E}_W[\mathbf{1}\{s \nleftrightarrow Q\}\eta(W \setminus C_Q)],$$
$$\chi_W(Q) = \mathbb{E}_W[\mathbf{1}\{s \nleftrightarrow Q\}\mathbf{1}\{R \nleftrightarrow Q\}\varrho(W \setminus C_Q)\eta(W \setminus C_Q)].$$

Write $\phi = \phi_U$ and $\chi = \chi_U$. Assume that, for all $W \subseteq U$ containing s and v and all $Q \subseteq W$,

$$\phi_W(Q)\,\mathbb{P}_W(v \nleftrightarrow X) \le \mathbb{P}_W(v \nleftrightarrow X \cup Q)\,\eta(W), \tag{31}$$

and assume that $Q \mapsto \phi_W(Q)$ is decreasing. Then

$$\mathbb{P}_U\big(R \nleftrightarrow X \cup N,\ R \leftrightarrow v\big)\,\phi(N_1) \le \mathbb{P}_U\big(v \nleftrightarrow X \cup N \cup N_1\big)\,\chi(N \cap N_1)\,. \tag{32}$$

Proof. We use strong induction on $|U|$. Suppose first that $N \cap N_1 = \varnothing$. Inequality (31), applied with $W = U$ and $Q = N_1$, gives

$$\phi(N_1)\mathbb{P}_U(v \nleftrightarrow X) \le \mathbb{P}_U(v \nleftrightarrow X \cup N_1)\eta(U)\,.$$

Inequality (30) gives

$$\mathbb{P}_U(R \nleftrightarrow X \cup N, R \leftrightarrow v)\mathbb{P}_U(v \nleftrightarrow X \cup N_1) \le \mathbb{P}_U(v \nleftrightarrow X \cup N \cup N_1)\mathbb{P}_U(R \nleftrightarrow X, R \leftrightarrow v)\,.$$

Multiply these inequalities. If $\mathbb{P}_U(v \nleftrightarrow X) > 0$, divide by this probability and use $\chi(\varnothing) = \varrho(U)\eta(U)$ to obtain (32). If $\mathbb{P}_U(v \nleftrightarrow X) = 0$, then

$$\mathbb{P}_U(R \nleftrightarrow X \cup N, R \leftrightarrow v) \le \mathbb{P}_U(v \nleftrightarrow X \cup N) \le \mathbb{P}_U(v \nleftrightarrow X) = 0\,.$$

The left side of (32) therefore vanishes in the zero-denominator case.

Now suppose that $Z = N \cap N_1$ is nonempty. The left side again vanishes if $v \in Z$, if $s \in Z$, or if $R \cap Z$ is nonempty. We may therefore assume that Z is disjoint from $R \cup \{s, v\}$. Put $U_0 = U \setminus Z$ and $X_0 = X \cap U_0$.

Expose all edges with exactly one endpoint in Z, and let $S \subseteq U_0$ consist of the vertices joined to Z by at least one exposed open edge. The events determining membership in S use disjoint edge families as the vertex varies. Thus S has a product law; write μ_Z for its mass function. For all $S_1, S_2 \subseteq U_0$,

$$\mu_Z(S_1)\mu_Z(S_2) = \mu_Z(S_1 \cap S_2)\mu_Z(S_1 \cup S_2)\,. \tag{33}$$

For $S \subseteq U_0$, write

$$\begin{aligned}
E(S) &= \mathbb{P}_{U_0}\big(R \nleftrightarrow X_0 \cup (N \setminus Z) \cup S,\ R \leftrightarrow v\big),\\
L(S) &= \phi_{U_0}\big((N_1 \setminus Z) \cup S\big),\\
M(S) &= \mathbb{P}_{U_0}\big(v \nleftrightarrow X_0 \cup (N \setminus Z) \cup (N_1 \setminus Z) \cup S\big),\\
K(S) &= \chi_{U_0}(S).
\end{aligned}$$

The domain Markov property gives four exact identities:

$$\begin{aligned}
\mathbb{P}_U(R \nleftrightarrow X \cup N, R \leftrightarrow v) &= \sum_{S \subseteq U_0} \mu_Z(S)E(S),\\
\phi_U(N_1) &= \sum_{S \subseteq U_0} \mu_Z(S)L(S),\\
\mathbb{P}_U(v \nleftrightarrow X \cup N \cup N_1) &= \sum_{S \subseteq U_0} \mu_Z(S)M(S),\\
\chi_U(Z) &= \sum_{S \subseteq U_0} \mu_Z(S)K(S).
\end{aligned}$$

These formulas express the four quantities in the percolation model on the smaller vertex set U_0.

It remains to verify the pointwise hypothesis of the four functions theorem. Fix $S_1, S_2 \subseteq U_0$, and put

$$Q = S_1 \cup \big((N \setminus Z) \setminus S_2\big), \qquad Q_1 = S_2 \cup \big((N_1 \setminus Z) \setminus S_1\big)\,.$$

Avoidance is decreasing in the avoided set, and $Q \mapsto \phi_{U_0}(Q)$ is decreasing. Hence

$$E(S_1) \le \mathbb{P}_{U_0}(R \not\leftrightarrow X_0 \cup Q, R \leftrightarrow v), \qquad L(S_2) \le \phi_{U_0}(Q_1)\,.$$

Because $Z = N \cap N_1$, the two constructed sets satisfy

$$Q \cup Q_1 = (N \setminus Z) \cup (N_1 \setminus Z) \cup S_1 \cup S_2, \qquad Q \cap Q_1 = S_1 \cap S_2\,.$$

The induction hypothesis in U_0 therefore yields

$$E(S_1)L(S_2) \le M(S_1 \cup S_2)K(S_1 \cap S_2)\,. \tag{34}$$

For $W \subseteq U_0$, every vertex of $X \setminus X_0$ lies outside the induced model and is isolated. Hence the events $\{v \not\leftrightarrow X\}$ and $\{v \not\leftrightarrow X_0\}$, and their unions with Q, coincide under $\mathbb{P}_W$. This transfers (31) to the smaller problem. Nonnegativity, the vanishing condition for η, and the monotonicity of ϕ_W restrict directly to U_0.

Apply the Ahlswede–Daykin four-functions theorem to

$$\mu_Z(S)E(S), \qquad \mu_Z(S)L(S), \qquad \mu_Z(S)M(S), \qquad \mu_Z(S)K(S)\,.$$

The pointwise hypothesis follows from (34) and (33). The four exact identities then turn its conclusion into (32). This completes the strong induction. □

This proves the required comparison when the two extra avoided sets overlap. We will use it after exposing C_Y, where the two avoided sets are equal.

Step 4: One-Vertex Comparison. Let $\eta_R(U) = \mathrm{Cov}_U(g(\mathcal{C}_s), \mathbf{1}\{R \leftrightarrow X\})$ and let $\eta_v(U)$ be as in (11). Then $\eta_v(U) \ge 0$ and

$$\varrho(U)\,\eta_v(U) \le \eta_R(U)\,. \tag{35}$$

Write J for $\{R \not\leftrightarrow X\} \cap \{R \leftrightarrow v\}$. The union of $\{R \leftrightarrow X\}$ and J is increasing, so Harris gives $\eta_R \ge \mathbb{E}_U[g(\mathcal{C}_s)]\mathbb{P}_U(J) - \mathbb{E}_U[g(\mathcal{C}_s); J]$. The function $g(\mathcal{C}_s)$ is increasing in $\mathcal{C}_X$ and does not depend on $\mathcal{C}_v$, while $\mathbf{1}_J$ is decreasing in $\mathcal{C}_X$ and increasing in $\mathcal{C}_v$. Hence (5) with $S_1 = X$ and $S_2 = \{v\}$ gives $\mathbb{P}_U(v \not\leftrightarrow X)\mathbb{E}_U[g(\mathcal{C}_s); J] \le \mathbb{P}_U(J)\mathbb{E}_U[g(\mathcal{C}_s); v \not\leftrightarrow X]$, and substituting the definitions gives (35).

Step 5: Exposing C_Y. Let $X = \{s\} \cup \{z_1, \ldots, z_\ell\}$, let v lie outside $X \cup Y$, and let R_0 be disjoint from $X \cup Y \cup \{v\}$. Write $U = V \setminus C_Y$ and

$$\mathcal{H} := \mathbb{E}\big[\mathbf{1}_D \mathbf{1}\{R_0 \not\leftrightarrow Y\}\,\eta_{R_0}(U)\big] - \tau(R_0)\,\mathbb{E}\big[\mathbf{1}_D\,\eta_v(U)\big]\,.$$

$\mathcal{H}$ is nonnegative. Indeed, for every $W \subseteq V$ and $Q \subseteq W$, the stopped-covariance estimate gives

$$\phi_W(Q)\mathbb{P}_W(v \not\leftrightarrow X) \le \mathbb{P}_W(v \not\leftrightarrow X \cup Q)\eta_v(W),$$

and $Q \mapsto \phi_W(Q)$ is decreasing. Thus Lemma 7.5 applies with $U = V$, $R = R_0$, $N = N_1 = Y$, and $\eta = \eta_v$. Since $B_{\ell+1} = X \cup Y$, the ratio $\mathbb{P}(H(R_0; \{v\}, X \cup Y))/\mathbb{P}(v \not\leftrightarrow X \cup Y)$ is $\tau(R_0)$. Dividing by the positive denominator gives $\tau(R_0)\mathbb{E}[\mathbf{1}_D\eta_v(U)] \le \mathbb{E}[\mathbf{1}_D\mathbf{1}\{R_0 \not\leftrightarrow Y\}\varrho(U)\eta_v(U)]$. Inequality (35) inside U bounds the integrand on the right.

Step 6: Exposing C_R. Let z be a vertex and let B contain $\{s\} \cup Y$ but not z, and abbreviate $H_R = H(R; \{z\}, B)$ and $E = \{z \not\leftrightarrow B\}$, with $\mathbb{P}(E) > 0$. Summing over the values of $\mathcal{C}_R$ gives $\mathbb{E}[\Theta; H_R] = -\mathbb{E}[\Psi(C_R); H_R]$: on H_R the set C_R is disjoint from B and every edge with exactly one endpoint in C_R is closed, so the conditional expectation of Θ is $-\Psi(C_R)$. On H_R at least one vertex of R is joined to z, so $C_z \subseteq C_R$, and therefore

$$\Delta(R) = \mathbb{E}\big[\Psi(C_R) - \Psi(C_z); H_R\big] \ge 0\,,$$

with $\Delta(\{u\}) = 0$ for every vertex u, since $C_{\{u\}} = C_z$ on $H_{\{u\}}$. Set

$$\begin{aligned}\Gamma_{z,B}(Q) = {} & \mathbb{P}(z \not\leftrightarrow B)\,\mathbb{E}[\Psi(C_z); H(Q; \{z\}, B)] \\ & - \mathbb{E}[\Psi(C_z); z \not\leftrightarrow B]\,\mathbb{P}(H(Q; \{z\}, B)).\end{aligned}$$

This is (25) with $s = z$, avoided set B, and $g(K) = \Psi(\{z\} \cup \operatorname{span} K)$. Expanding $\Gamma_{z,B}$, then using the displayed identity, the definition of Δ and (10), gives

$$\mathbb{E}[\Theta; H_R] - \pi(R)\,\mathbb{E}[\Theta; E] = -\frac{\Gamma_{z,B}(R)}{\mathbb{P}(E)} - \Delta(R), \qquad \pi(R) = \frac{\mathbb{P}(H_R)}{\mathbb{P}(E)}\,, \tag{36}$$

and hence, dropping $\Delta(R) \geq 0$,

$$\mathbb{E}[\Theta; H_R] - \pi(R)\,\mathbb{E}[\Theta; E] \leq -\frac{\Gamma_{z,B}(R)}{\mathbb{P}(E)}\,, \tag{37}$$

with equality whenever R is a single vertex.

Step 7: Auxiliary Vertices. For each increasing function q of edge sets, let $\Theta_q, \Psi_q, \mathcal{H}(q), \Gamma^q_{z,B}$, and $\Delta^q_{z,B}$ denote the quantities defined in (9) and Steps 5–6 with g replaced by q. Without subscripts, Γ^q denotes the function in (25) with starting vertex s, avoided set Y, and $g = q$. Put

$$\mathcal{W}(q) = \mathbb{E}\Big[\Theta_q\,\mathfrak{M}\big[R \mapsto \mathbf{1}_{H(R;\{s\},Y)}\big]\Big],$$

where $\mathfrak{M}$ acts on the argument R while $\mathcal{C}_s$ and $\mathcal{C}_Y$ are held fixed. For $1 \leq j \leq \ell$, let $\mathfrak{M}_j$ be the functional formed from $z_{j+1}, \ldots, z_\ell$, from v, and from R_0, with base avoided set B_{j+1}. In the term indexed by j, let $\Gamma^q_{z_j,B_j}$ be (25) formed with the increasing function

$$K \longmapsto \Psi_q(\{z_j\} \cup \operatorname{span} K).$$

Let $\Delta^q_j(R)$ be the quantity $\Delta(R)$ in Step 6 with $z = z_j$, $B = B_j$, and $\Psi = \Psi_q$.

Apply (36) first at z_1, then at z_2, and continue through z_ℓ. Linearity of $\mathfrak{M}$ gives

$$\mathcal{W}(q) = \mathcal{H}(q) + \sum_{j=1}^{\ell} \frac{\mathfrak{M}_j\big[\Gamma^q_{z_j,B_j}\big]}{\mathbb{P}(z_j \nleftrightarrow B_j)} + \sum_{j=1}^{\ell} \Delta^q_j(R_0)\,. \tag{38}$$

At the jth subtraction, (36) produces one covariance with the shorter list and one copy of Δ^q_j. Step 1 writes the remaining functional as evaluation at R_0, with coefficient one, plus evaluations at singletons. Because Δ^q_j vanishes at every singleton, only $\Delta^q_j(R_0)$ remains. The term depending only on $\mathcal{C}_Y$ has zero expectation because $\mathbb{E}[\Theta_q \mid \mathcal{C}_Y] = 0$. After the last subtraction, the remaining term is $\mathcal{H}(q)$, proving (38).

We next pass from $\mathcal{W}(q)$ to the covariance in (25). For possible values K_s and K_Y of $\mathcal{C}_s$ and $\mathcal{C}_Y$, respectively, define

$$h(K_s, K_Y) = \mathfrak{M}[R \mapsto \mathbf{1}\{R \cap (\{s\} \cup \operatorname{span} K_s) \neq \varnothing,\ R \cap (Y \cup \operatorname{span} K_Y) = \varnothing\}]\,.$$

If $K_Y \subseteq L_Y$, then $h(K_s, L_Y) \leq h(K_s, K_Y)$. Indeed, the term evaluated at R_0 requires R_0 to avoid the larger cluster represented by L_Y. Each singleton term is independent of the second argument because, on $D = \{s \nleftrightarrow Y\}$, a vertex connected to s cannot also be connected to Y.

For an increasing function q of edge sets, put

$$\mathcal{C}(q) = \mathbb{P}(D)\,\mathbb{E}\big[q(\mathcal{C}_s)h(\mathcal{C}_s, \mathcal{C}_Y); D\big] - \mathbb{E}\big[q(\mathcal{C}_s); D\big]\,\mathbb{E}\big[h(\mathcal{C}_s, \mathcal{C}_Y); D\big].$$

Thus $\mathcal{C}(q)$ is the unnormalized covariance under $\mathbb{P}(\,\cdot \mid D)$. By definition, it equals $\mathfrak{M}[\Gamma^q]$.

We now define the alternating conditional-expectation operator used below. Let $\nu = \mathbb{P}(\,\cdot \mid D)$ and, for every value L of $\mathcal{C}_Y$ in the support of ν, set

$$Q_q(L) = \mathbb{E}_\nu[q(\mathcal{C}_s) \mid \mathcal{C}_Y = L].$$

For every value K of $\mathcal{C}_s$ in the support of ν, define

$$(\mathcal{K}q)(K) = \mathbb{E}_\nu[Q_q(\mathcal{C}_Y) \mid \mathcal{C}_s = K]. \tag{39}$$

This finite-state formula first applies conditional expectation given $\mathcal{C}_Y$ and then conditional expectation given $\mathcal{C}_s$; it determines $\mathcal{K}$ without any choice of versions on null states.

For completeness, define the reverse conditional-covariance term by

$$\mathcal{W}_2(q) = \mathbb{P}(D)\,\mathbb{E}_\nu[\mathrm{Cov}_\nu(Q_q(\mathcal{C}_Y), h(\mathcal{C}_s, \mathcal{C}_Y) \mid \mathcal{C}_s)]\,.$$

The definition of Θ_q gives

$$\mathcal{W}(q) = \mathbb{P}(D)\,\mathbb{E}_\nu[\mathrm{Cov}_\nu(q(\mathcal{C}_s), h(\mathcal{C}_s, \mathcal{C}_Y) \mid \mathcal{C}_Y)]\,.$$

Twice applying the law of total covariance, or directly expanding the finite sums, yields

$$\mathcal{C}(q) - \mathcal{C}(\mathcal{K}q) = \mathbb{P}(D)\big(\mathcal{W}(q) + \mathcal{W}_2(q)\big). \tag{40}$$

If q is increasing, then

$$L \longmapsto \mathbb{E}_\nu[q(\mathcal{C}_s) \mid \mathcal{C}_Y = L]$$

is decreasing: enlarging the exposed cluster of Y closes more edges available to C_s. The function $L \mapsto h(K, L)$ is also decreasing. Conditional positive association therefore gives $\mathcal{W}_2(q) \geq 0$. The same cluster-exploration coupling shows that $\mathcal{K}q$ is increasing. Consequently, once $\mathcal{W}(q) \geq 0$ is known for every increasing q, (40) gives

$$\mathcal{C}(\mathcal{K}q) \leq \mathcal{C}(q).$$

It remains to show that this monotonicity forces $\mathcal{C}(q) \geq 0$. For a function on a finite state space, write $\mathrm{osc}(q) = \max q - \min q$. Set

$$\varepsilon = \prod_{e\colon e\cap Y \neq \varnothing} (1 - p_e) > 0, \qquad \beta = 1 - \varepsilon < 1.$$

During the first conditional draw in (39), all edges meeting Y are closed with conditional probability at least ε. On that event $\mathcal{C}_Y = \varnothing$, and the law of the second draw of $\mathcal{C}_s$ is independent of the initial value K. Thus every transition law of $\mathcal{K}$ has a common component of mass at least ε, and the elementary coupling bound gives

$$\mathrm{osc}(\mathcal{K}^n q) \leq \beta^n\,\mathrm{osc}(q).$$

If $H_0 = \max_{K,L} |h(K, L)|$, then

$$|\mathcal{C}(\mathcal{K}^n q)| \leq 2\mathbb{P}(D)^2 H_0\ \mathrm{osc}(\mathcal{K}^n q) \leq 2H_0\beta^n\,\mathrm{osc}(q).$$

Iteration gives $\mathcal{C}(\mathcal{K}^n q) \leq \mathcal{C}(q)$, while the displayed bound tends to zero. Hence $\mathcal{C}(q) \geq 0$.

We can now close the strong induction on ℓ. We prove simultaneously, for every increasing q, that $\mathcal{W}(q) \geq 0$ and $\mathfrak{M}[\Gamma^q] \geq 0$. If $\ell = 0$, then (38) reduces to $\mathcal{W}(q) = \mathcal{H}(q)$. Step 5 gives $\mathcal{H}(q) \geq 0$, and the preceding covariance argument gives $\mathfrak{M}[\Gamma^q] \geq 0$.

For the inductive step, suppose both conclusions hold for all shorter lists. The term $\mathcal{H}(q)$ in (38) is nonnegative by Step 5. Each denominator $\mathbb{P}(z_j \nleftrightarrow B_j)$ is positive because closing all edges incident to z_j forces $z_j \nleftrightarrow B_j$ and has positive probability. The function used in $\Gamma^q_{z_j,B_j}$ is increasing, and its auxiliary list is $z_{j+1}, \ldots, z_\ell$. The induction hypothesis gives $\mathfrak{M}_j[\Gamma^q_{z_j,B_j}] \geq 0$. Step 6 gives $\Delta^q_j(R_0) \geq 0$. Every term on the right side of (38) is therefore nonnegative, so $\mathcal{W}(q) \geq 0$. The covariance argument then gives $\mathfrak{M}[\Gamma^q] \geq 0$ at length ℓ. The first conclusion at length ℓ uses only the second conclusion for shorter lists, while the second uses the first at the current length. Thus the induction is not circular.

Step 8: Induction on T. Recall Y, F, T, r, and the numbers m_x of Section 4. For a finite set R of vertices put

$$J_x(R) = \{R \nleftrightarrow Y\} \cap \{R \leftrightarrow x\} \cap \bigcap_{y\in T,\ r(y)<r(x)} \{R \nleftrightarrow y\},$$

$$\Lambda_R(T) = \mathbb{E}\big[F(C_R);\, R \nleftrightarrow Y,\ R \leftrightarrow T\big] - \sum_{x\in T} \mathbb{P}\big(J_x(R)\big)\, m_x\,.$$

For $R = \{u\}$ these are (14) and (15).

We can now prove the version of Theorem 4.1 with a finite starting set.

Theorem 7.6. *Let V be a finite vertex set with edge probabilities in $[0,1]$. Let $Y, R \subseteq V$, and let $T \subseteq V \setminus Y$ be finite. Assume that R is nonempty and disjoint from $Y \cup T$. Let F be an increasing real function on the subsets of V, and assume $\mathbb{P}(x \nleftrightarrow Y) > 0$ for every $x \in T$. For each $x \in T$, let*

$$m_x = \frac{\mathbb{E}[F(C_x);\, x \nleftrightarrow Y]}{\mathbb{P}(x \nleftrightarrow Y)}\,.$$

Let r be an injective real function on T such that $m_x \leq m_y$ whenever $r(x) < r(y)$. For $x \in T$, define

$$J_x(R) = \{R \nleftrightarrow Y, R \leftrightarrow x\} \cap \bigcap_{\substack{y \in T \\ r(y) < r(x)}} \{R \nleftrightarrow y\},$$

and set

$$\Lambda_R(T) = \mathbb{E}[F(C_R); R \nleftrightarrow Y, R \leftrightarrow T] - \sum_{x \in T} \mathbb{P}(J_x(R)) m_x.$$

Then $\Lambda_R(T) \geq 0$.

Proof. If R is a singleton, the conclusion is Theorem 4.1. Assume first that R has at least two elements and that every edge probability lies strictly between zero and one. We use two inductions. The first establishes the transfer estimate (41), which is used in the main induction.

For each $S \subseteq T$, let $\mathsf{P}(S)$ assert that (41), with S in place of T, holds for every nonempty R with $|R| \geq 2$ and $R \cap (Y \cup S) = \varnothing$, every length ℓ, every pairwise distinct list $z_1, \ldots, z_\ell$ outside $R \cup Y \cup S$, and every vertex v outside $R \cup Y \cup S \cup \{z_1, \ldots, z_\ell\}$. In this definition, $\mathfrak{M}$ is given by (26)–(27) with initial avoided set $B_1 = Y \cup S$. We prove $\mathsf{P}(S)$ by induction on $|S|$. Thus, for the current set T, the conclusion is

$$\mathfrak{M}\big[Q \mapsto \Lambda_Q(T)\big] \geq 0\,. \tag{41}$$

The claim is immediate for $S = \varnothing$. For the inductive step, fix $R, z_1, \ldots, z_\ell, v$ satisfying these conditions, choose $k \in S$ with maximal $r(k)$, and write $S_0 = S \setminus \{k\}$. To lighten notation, write $T = S$ and $T_0 = S_0$ for the remainder of this induction step. Then $m_x \leq m_k$ for every $x \in T_0$. Write

$$E_k = \{k \nleftrightarrow Y \cup T_0\}, \qquad G_k(Q) = H(Q; \{k\}, Y \cup T_0), \qquad d_k = \mathbb{P}(E_k),$$

and define

$$K_k(Q) = \mathbb{E}[F(C_k) - m_k;\, G_k(Q)], \qquad c_k(Q) = \frac{\mathbb{P}(G_k(Q))}{d_k},$$

$$\kappa_k = m_k d_k - \mathbb{E}[F(C_k); E_k]\,.$$

The all-closed configuration belongs to E_k, so $d_k > 0$.

Adding the vertex k, whose value $r(k)$ is maximal, to T_0 gives

$$\Lambda_Q(T) = \Lambda_Q(T_0) + \mathbb{E}[F(C_Q) - m_k;\, G_k(Q)]\,.$$

On $G_k(Q)$ we have $C_k \subseteq C_Q$. Monotonicity of F therefore gives the lower bound

$$\Lambda_Q(T) \geq \Lambda_Q(T_0) + K_k(Q),$$

with equality whenever Q is a singleton. Step 1 now gives

$$\mathfrak{M}[\Lambda.(T)] \geq \mathfrak{M}[\Lambda.(T_0) + K_k]\,. \tag{42}$$

Let $g_k(K) = F(\{k\} \cup \operatorname{span} K)$ and define

$$\Gamma_k(Q) = d_k \mathbb{E}[g_k(\mathcal{C}_k); G_k(Q)] - \mathbb{E}[g_k(\mathcal{C}_k); E_k]\mathbb{P}(G_k(Q)).$$

This is (25) with starting vertex k and avoided set $Y \cup T_0$. Direct expansion gives, for every vertex set Q,

$$d_k K_k(Q) = \Gamma_k(Q) - \kappa_k d_k c_k(Q)\,. \tag{43}$$

Step 7 gives $\mathfrak{M}[\Gamma_k] \ge 0$. For the second application, set $h_k(K) = \mathbf{1}\{K \ne \varnothing\}$ and $q_k = \mathbb{P}(E_k, \mathcal{C}_k = \varnothing) > 0$. At every set Q where $\mathfrak{M}$ evaluates a function, $k \notin Q$; hence $G_k(Q)$ implies $\mathcal{C}_k \ne \varnothing$. The version of Γ_k formed with h_k is therefore

$$d_k\mathbb{P}(G_k(Q)) - (d_k - q_k)\mathbb{P}(G_k(Q)) = q_k d_k c_k(Q).$$

Step 7 gives $q_k d_k \mathfrak{M}[c_k] \ge 0$, and thus $\mathfrak{M}[c_k] \ge 0$.

Partitioning $\{k \nleftrightarrow Y\}$ according to the first vertex of T_0 connected to k gives

$$\Lambda_{\{k\}}(T_0) - \kappa_k = \sum_{x \in T_0} \mathbb{P}(J_x(\{k\}))(m_k - m_x) \ge 0\,. \tag{44}$$

The induction hypothesis for T_0, with k inserted before $z_1, \dots, z_\ell$ in the auxiliary list, is

$$\mathfrak{M}[\Lambda.(T_0)] - \Lambda_{\{k\}}(T_0)\mathfrak{M}[c_k] \ge 0\,.$$

Combining (42), (43), (44), and the inserted-list induction hypothesis gives

$$\begin{aligned} d_k\mathfrak{M}[\Lambda.(T)] &\ge d_k\mathfrak{M}[\Lambda.(T_0) + K_k] \\ &= d_k\mathfrak{M}[\Lambda.(T_0)] + \mathfrak{M}[\Gamma_k] - \kappa_k d_k \mathfrak{M}[c_k] \\ &\ge d_k\big(\mathfrak{M}[\Lambda.(T_0)] - \Lambda_{\{k\}}(T_0)\mathfrak{M}[c_k]\big) + \mathfrak{M}[\Gamma_k] \ge 0. \end{aligned}$$

Since $d_k > 0$, this proves (41).

Apply (41) to T_0, with no auxiliary vertices and with comparison vertex k. The definition of $\mathfrak{M}$ gives

$$\mathbb{P}\big(G_k(R)\big)\Lambda_{\{k\}}(T_0) \le d_k\Lambda_R(T_0)\,. \tag{45}$$

We now prove $\Lambda_R(T) \ge 0$ by induction on $|T|$. The claim is immediate for $T = \varnothing$. For the inductive step, choose $k \in T$ with maximal $r(k)$, put $T_0 = T \setminus \{k\}$, and define

$$E_k = \{k \nleftrightarrow Y \cup T_0\}, \quad G_k(R) = H(R; \{k\}, Y \cup T_0), \quad d_k = \mathbb{P}(E_k),$$

$$\kappa_k = m_k d_k - \mathbb{E}[F(C_k); E_k].$$

Also put

$$I_k = \mathbb{E}[F(C_R) - m_k;\, G_k(R)]\,.$$

The exact decomposition after adjoining k is

$$\Lambda_R(T) = \Lambda_R(T_0) + I_k\,. \tag{46}$$

We next estimate I_k. Under the conditional law on E_k, apply positive association to the increasing random variables

$$F(C_k) \quad \text{and} \quad \alpha_{R, Y\cup T_0}(C_k).$$

The identity (29), first alone and then with the factor $F(C_k)$, gives

$$\mathbb{E}[\alpha_{R,Y\cup T_0}(C_k); E_k] = \mathbb{P}(G_k(R))$$

and

$$\mathbb{E}[F(C_k)\alpha_{R,Y\cup T_0}(C_k); E_k] = \mathbb{E}[F(C_k); G_k(R)]\,.$$

Conditional positive association gives

$$d_k\mathbb{E}[F(C_k); G_k(R)] \ge \mathbb{E}[F(C_k); E_k]\mathbb{P}(G_k(R)).$$

Because $C_k \subseteq C_R$ on $G_k(R)$, monotonicity of F then gives

$$d_k I_k \ge -\mathbb{P}(G_k(R))\kappa_k\,. \tag{47}$$

By (44), $\kappa_k \le \Lambda_{\{k\}}(T_0)$. Therefore

$$\begin{aligned} d_k\Lambda_R(T) &= d_k\Lambda_R(T_0) + d_k I_k \\ &\ge d_k\Lambda_R(T_0) - \mathbb{P}(G_k(R))\kappa_k \\ &\ge d_k\Lambda_R(T_0) - \mathbb{P}(G_k(R))\Lambda_{\{k\}}(T_0) \ge 0, \end{aligned}$$

where the last inequality is (45). Dividing by $d_k > 0$ completes the induction for edge probabilities strictly between zero and one.

It remains to allow edge probabilities equal to zero or one. For every injective r satisfying $m_x \le m_y$ whenever $r(x) < r(y)$, the partition by the first connected vertex gives

$$\Lambda_R(T) = \mathbb{E}\left[\mathbf{1}\{R \nleftrightarrow Y,\ R \leftrightarrow T\}\left(F(C_R) - \min_{x \in T \cap C_R} m_x\right)\right].$$

This expression is independent of the order used to break ties. It depends continuously on the edge probabilities whenever $\mathbb{P}(x \nleftrightarrow Y) > 0$ for every $x \in T$. Approximate the given probabilities by probabilities in the open unit interval, and for each approximation choose an injective r ordered by its corresponding values m_x. The interior case applies to every approximation, so continuity gives $\Lambda_R(T) \ge 0$ at the original probabilities. □

Step 9: Removing the Conditioning. An identity and a comparison relate F to Φ_a when several vertices replace one. Let R be a nonempty finite set of vertices, let a be a vertex and let $\mathcal{S}$ be a family of edge sets.

Lemma 7.7. *For every nonempty set R, every vertex a, every increasing function F of vertex sets, and every family $\mathcal{S}$ of edge sets,*

$$\mathbb{E}\big[F(C_R) - F(C_a);\ R \nleftrightarrow a,\ \mathcal{C}_R \in \mathcal{S}\big] = \mathbb{E}\big[\Phi_a(C_R);\ R \nleftrightarrow a,\ \mathcal{C}_R \in \mathcal{S}\big]. \tag{48}$$

Proof. Condition on $\mathcal{C}_R$. On $\{R \nleftrightarrow a\}$ the vertex a lies outside C_R, every edge with exactly one endpoint in C_R is closed, and the edges with both endpoints outside C_R retain their original probabilities. The conditional expectation of $F(C_a)$ is therefore the expectation subtracted from $F(C_R)$ in the definition of $\Phi_a(C_R)$. Both $\{R \nleftrightarrow a\}$ and $\{\mathcal{C}_R \in \mathcal{S}\}$ are determined by $\mathcal{C}_R$. Summing over the possible values of $\mathcal{C}_R$ proves (48). □

The next lemma handles the case where the starting set already contains a vertex whose unconditional mean is at least that of a.

Lemma 7.8. *Let R be a nonempty set of vertices, let a be a vertex, and let F be an increasing function of vertex sets. If there is a vertex $s \in R$ such that $\mathbb{E}[F(C_a)] \le \mathbb{E}[F(C_s)]$, then*

$$\mathbb{E}\big[F(C_R) - F(C_a);\ R \nleftrightarrow a\big] \ge 0. \tag{49}$$

Proof. For $K = C_s$, the function in (28) is

$$\alpha_{R,\{a\}}(C_s) = \mathbb{P}_{V \setminus C_s}(R \setminus C_s \nleftrightarrow a)$$

on $\{s \nleftrightarrow a\}$, because $s \in R \cap C_s$. It is increasing, and (29) gives

$$\mathbb{E}[\Phi_a(C_s); R \nleftrightarrow a] = \mathbb{E}[\Phi_a(C_s)\alpha_{R,\{a\}}(C_s); s \nleftrightarrow a].$$

Apply (6) with vertex s, with $X = \{a\}$ and with the two increasing functions $\mathcal{C} \mapsto \Phi_a(\{s\} \cup \operatorname{span} \mathcal{C})$ and $\mathcal{C} \mapsto \alpha_{R,\{a\}}(\{s\} \cup \operatorname{span} \mathcal{C})$. Lemma 5.1(ii) gives

$$\mathbb{E}[\Phi_a(C_s); s \nleftrightarrow a] = \mathbb{E}[F(C_s)] - \mathbb{E}[F(C_a)] \ge 0.$$

Since the other factor in (6) is nonnegative, the preceding identity gives $\mathbb{E}[\Phi_a(C_s); R \nleftrightarrow a] \ge 0$. On $\{R \nleftrightarrow a\}$, the inclusion $C_s \subseteq C_R$ and monotonicity of Φ_a imply $\mathbb{E}[\Phi_a(C_R); R \nleftrightarrow a] \ge 0$. Equation (48) now gives (49). □

Proof of Theorem 7.1. Put $T = A \setminus \{a\}$. If $a \in R$, then $\{R \nleftrightarrow a\}$ is empty and the restricted expectation in (22) is zero. If $R \cap T \ne \varnothing$, choose $s \in R \cap T$; then $\{R \leftrightarrow T\}$ occurs surely and (49) applies. Otherwise R is disjoint from A. Let T_1 be the set of $x \in T$ with $\mathbb{P}(x \nleftrightarrow a) > 0$, and apply Theorem 7.6 with $Y = \{a\}$, with T_1, with Φ_a and with R. By Lemma 5.1(ii) the number (13) attached to $x \in T_1$ is

$(\mathbb{E}[F(C_x)] - \mathbb{E}[F(C_a)])/\mathbb{P}(x \nleftrightarrow a) \geq 0$, so $\mathbb{E}[\Phi_a(C_R); R \nleftrightarrow a, R \leftrightarrow T_1] \geq 0$; and (48) converts this into (22) with T_1 in place of $A \setminus \{a\}$. The symmetric difference between the restricting events is null: indeed, a configuration in $\{R \nleftrightarrow a, R \leftrightarrow T\} \setminus \{R \leftrightarrow T_1\}$ connects R to some $x \in T \setminus T_1$ while $R \nleftrightarrow a$, and therefore lies in the null event $\{x \nleftrightarrow a\}$. A finite union of null events is null, so the two restricted expectations agree. □

The same comparison has a second application. By exposing the edges incident to o, we can represent C_o as a union of clusters in the measure used to choose the minimizer in Question 9.

8 Question 9

Question 9 selects a using probabilities computed with every edge incident to o closed, and then asks for (4) in the original measure. Exposing the edges at o turns C_o into the union of the clusters of a random vertex set. Theorem 7.1 applies for every realization of that set; Figure 3 depicts the construction.

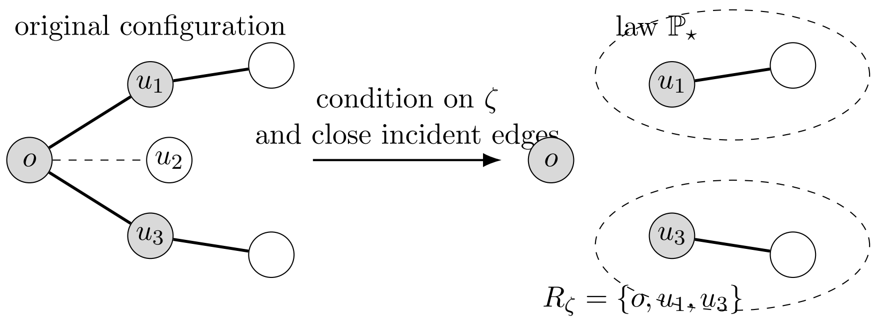

Figure 3: Fix the complete state ζ of the edges incident to o. Record the endpoints of its open edges in R_ζ, then delete every edge incident to o. For the same nonincident-edge configuration, C_o in the original graph is exactly the union C_{R_ζ} in the graph with those edges deleted. The vertices of R_ζ need not be connected, and their clusters need not be disjoint. The layout is schematic.

Theorem 8.1. *Let A be a nonempty finite set of vertices, let o and b be vertices, and let $a \in A$ satisfy $\mathbb{P}_\star(a \leftrightarrow b) \leq \mathbb{P}_\star(x \leftrightarrow b)$ for every $x \in A$. Then, with both probabilities below evaluated under the original measure,*

$$\mathbb{P}(o \leftrightarrow b, o \leftrightarrow A) \geq \mathbb{P}(o \leftrightarrow A, a \leftrightarrow b).$$

Thus Question 9 has an affirmative answer for every minimizer.

Proof. Condition on the states of the edges incident to o. For an incident-edge configuration ζ, let R_ζ be the union of $\{o\}$ and the set of vertices $u \neq o$ with $\{o, u\}$ open in ζ. The nonincident edges have exactly their law under $\mathbb{P}_\star$, and they are independent of ζ.

Any open walk may be reduced to a simple path. Unless its endpoint is o, a simple path from o uses exactly one edge incident to o, as its first edge; the remaining path lies in the graph with those incident edges deleted. Conversely, if $u \in R_\zeta \setminus \{o\}$, every path in the deleted-edge graph beginning at u can be preceded by the open edge $\{o, u\}$; the isolated starting vertex o contributes only o. Hence, pointwise in the nonincident-edge configuration,

$$C_o = C_{R_\zeta}, \qquad \{o \leftrightarrow A\} = \{R_\zeta \leftrightarrow A\}, \tag{50}$$

where C_{R_ζ} and $\{R_\zeta \leftrightarrow A\}$ are computed under $\mathbb{P}_\star$, for which o is isolated and therefore contributes the single vertex o to the union. Moreover, on $\{o \nleftrightarrow a\}$ no path from a can use an edge incident to o. Hence C_a in the original configuration equals the cluster computed after deleting the incident edges.

Put $F = \mathbf{1}\{b \in \cdot\}$. On $\{o \leftrightarrow a\}$ the clusters of o and a coincide, so

$$\mathbb{P}(o \leftrightarrow b, o \leftrightarrow A) - \mathbb{P}(a \leftrightarrow b, o \leftrightarrow A) = \mathbb{E}\big[F(C_o) - F(C_a);\ o \nleftrightarrow a,\ o \leftrightarrow A \setminus \{a\}\big]\,,$$

and by (50) the right side equals

$$\sum_{\zeta} \mathbb{P}(\zeta)\, \mathbb{E}_\star\big[F(C_{R_\zeta}) - F(C_a);\ R_\zeta \nleftrightarrow a,\ R_\zeta \leftrightarrow A \setminus \{a\}\big]\,,$$

the events and clusters inside the expectation being computed under $\mathbb{P}_\star$. The hypothesis on a says $\mathbb{E}_\star[F(C_a)] \le \mathbb{E}_\star[F(C_x)]$ for every $x \in A$, so each summand is nonnegative by Theorem 7.1 applied under $\mathbb{P}_\star$ with the set R_ζ. Summing over ζ gives (4). No division is used, so states ζ of probability zero and coincidences among o, a, b and the vertices of A are covered. □

The incident-edge decomposition does not apply to Question 8, whose minimizer is selected on $\{o \nleftrightarrow A\}$. We now give a counterexample to the proposed inequality.

9 Question 8

Set

$$\rho_x = \mathbb{P}(x \leftrightarrow b,\ o \nleftrightarrow A) \qquad \text{for } x \in A\,.$$

The phrase *let a minimize* is understood to assert the conclusion for an arbitrary minimizer. The choice matters when the minimum is attained more than once. The next example lies on the boundary $\mathbb{P}(o \nleftrightarrow A) = 0$: both vertices of A minimize ρ, but one of them violates (4). Thus Question 8, read as a statement about every minimizer, is false. The example does not address an existential tie-breaking interpretation.

Proposition 9.1. *There is a four-vertex weighted graph, a two-element set A, and vertices o and b outside A with the following property. Both elements of A minimize $\mathbb{P}(x \leftrightarrow b,\ o \nleftrightarrow A)$, but one minimizing vertex a satisfies*

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) < \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)\,.$$

Proof. Let $V = \{o, a_1, a_2, b\}$ and $A = \{a_1, a_2\}$. Set $p_{\{o,a_1\}} = p_{\{a_2,b\}} = 1$, $p_{\{a_1,a_2\}} = \frac{1}{2}$, and all other edge probabilities to zero. Since $\{o, a_1\}$ is open with probability one and $a_1 \in A$, the event $\{o \leftrightarrow A\}$ has probability one, so $\rho_{a_1} = \rho_{a_2} = 0$ and both vertices minimize. The event $\{a_2 \leftrightarrow b\}$ has probability one, so $\mathbb{P}(o \leftrightarrow A,\ a_2 \leftrightarrow b) = 1$. The event $\{o \leftrightarrow b\}$ requires the edge $\{a_1, a_2\}$ to be open, so $\mathbb{P}(o \leftrightarrow b, o \leftrightarrow A) = \frac{1}{2}$. Thus (4) fails for $a = a_2$. □

The counterexample already has two intermediate vertices, so even the restriction $|A| = 2$ does not make (4) valid. For completeness, we record two statements that do hold when $|A| \le 2$: a unique minimizer works, and every minimizer works if $\mathbb{P}(o \nleftrightarrow A) > 0$. We first cancel the contribution from $\{o \leftrightarrow a\}$.

Lemma 9.2. *Let A be a nonempty finite set of vertices, let $a \in A$, and let b and o be vertices. Then*

$$\mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b) - \mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) = \mathbb{P}\big(a \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow A \setminus \{a\}\big) - \mathbb{P}\big(o \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow A \setminus \{a\}\big)\,.$$

Proof. Split $\{o \leftrightarrow A\}$ into $\{o \leftrightarrow a\}$ and $\{o \nleftrightarrow a\} \cap \{o \leftrightarrow A \setminus \{a\}\}$. On $\{o \leftrightarrow a\}$ the clusters of o and a coincide, so the events $\{a \leftrightarrow b\}$ and $\{o \leftrightarrow b\}$ agree there and the two contributions from that part cancel. □

We use the following consequence of the BHK two-cluster inequality.

Lemma 9.3. *Let S_1 and S_2 be sets of vertices, let g_3 and g_4 be nonnegative real functions of two edge sets, each increasing in the first argument and decreasing in the second, and let h_1 and h_2 be nonnegative real functions of two edge sets, each decreasing in the first argument and increasing in the second. Let $D = \{S_1 \nleftrightarrow S_2\}$ and define $G_i(\omega) = g_i(\mathcal{C}_{S_1}(\omega), \mathcal{C}_{S_2}(\omega))$ and $H_i(\omega) = h_i(\mathcal{C}_{S_1}(\omega), \mathcal{C}_{S_2}(\omega))$. Then*

$$\mathbb{E}[G_3H_1; D]\,\mathbb{E}[G_4H_2; D] \le \mathbb{E}[G_3G_4; D]\,\mathbb{E}[H_1H_2; D]\,.$$

Proof. If $\mathbb{P}(D) = 0$, then both sides vanish, so assume $\mathbb{P}(D) > 0$. Theorem 3.2 applies to G_3, G_4 and, after exchanging S_1 and S_2, to H_1, H_2, giving

$$\mathbb{E}[G_3; D]\,\mathbb{E}[G_4; D] \le \mathbb{P}(D)\mathbb{E}[G_3G_4; D], \quad \mathbb{E}[H_1; D]\,\mathbb{E}[H_2; D] \le \mathbb{P}(D)\mathbb{E}[H_1H_2; D].$$

Inequality (5) applies to G_3, H_1 and to G_4, H_2, giving

$$\mathbb{P}(D)\mathbb{E}[G_3H_1; D] \le \mathbb{E}[G_3; D]\mathbb{E}[H_1; D], \quad \mathbb{P}(D)\mathbb{E}[G_4H_2; D] \le \mathbb{E}[G_4; D]\mathbb{E}[H_2; D].$$

All eight quantities are nonnegative. Multiplying the two inequalities of the second display and then using the first display gives the asserted inequality after division by $\mathbb{P}(D)^2$. ☐

Lemma 9.3 controls the configurations that remain after the cancellation in Lemma 9.2.

Theorem 9.4. *Let o and b be vertices and let A be a nonempty set of vertices with at most two elements. If $a \in A$ is the unique vertex minimizing $\mathbb{P}(x \leftrightarrow b,\ o \nleftrightarrow A)$ over $x \in A$, then*

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \ge \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)\,.$$

If $\mathbb{P}(o \nleftrightarrow A) > 0$, then

$$\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) \ge \mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)$$

for every $a \in A$ that realizes the minimum.

Proof. We first treat $|A| = 1$, then a strict minimum for $A = \{a, x\}$, and finally a tied minimum under the assumption $\mathbb{P}(o \nleftrightarrow A) > 0$.

If $A = \{a\}$, then on $\{o \leftrightarrow a\}$ the clusters of o and a coincide, so $\mathbb{P}(o \leftrightarrow A, a \leftrightarrow b) = \mathbb{P}(o \leftrightarrow a, o \leftrightarrow b) = \mathbb{P}(o \leftrightarrow b, o \leftrightarrow A)$, and both conclusions hold with equality.

Next let $A = \{a, x\}$ with $\rho_a < \rho_x$. By Lemma 9.2, and because the clusters of o and x coincide on $\{o \leftrightarrow x\}$, the inequality (4) to be proved reads

$$\mathbb{P}(a \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow x) \le \mathbb{P}(x \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow x)\,. \tag{51}$$

Apply Lemma 9.3 with $S_1 = \{a\}$ and $S_2 = \{o, x\}$, and write D for the event $\{a \nleftrightarrow o,\ a \nleftrightarrow x\}$. For edge sets K, L, let $g_3(K, L)$ indicate that a is connected to b in (V, K), and let $g_4(K, L)$ indicate that o is not connected to x in (V, L). Let $h_1(K, L)$ and $h_2(K, L)$ indicate, respectively, that o is connected to x and that x is connected to b in (V, L). Then g_3, g_4 are increasing in K and decreasing in L, while h_1, h_2 are decreasing in K and increasing in L. Evaluated at $(\mathcal{C}_a, \mathcal{C}_{\{o,x\}})$ on D, they are the four event indicators just described.

On $\{o \leftrightarrow x\}$ the clusters of o and x coincide, so $a \nleftrightarrow o$ is equivalent to $a \nleftrightarrow x$. Thus $D \cap \{o \leftrightarrow x\} = \{o \nleftrightarrow a,\ o \leftrightarrow x\}$, and

$$\mathbb{E}[g_3h_1; D] = \mathbb{P}(a \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow x), \qquad \mathbb{E}[h_1h_2; D] = \mathbb{P}(x \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow x)\,.$$

On the other hand $D \cap \{o \nleftrightarrow x\} = \{o \nleftrightarrow A\} \cap \{a \nleftrightarrow x\}$. Two vertices connected to b are connected to each other. Thus, on $D \cap \{o \nleftrightarrow x\}$, $\{a \leftrightarrow b\}$ implies $x \nleftrightarrow b$, and $\{x \leftrightarrow b\}$ implies $a \nleftrightarrow b$. Therefore

$$\mathbb{E}[g_3g_4; D] = \mathbb{P}(o \nleftrightarrow A,\ a \leftrightarrow b,\ x \nleftrightarrow b), \qquad \mathbb{E}[g_4h_2; D] = \mathbb{P}(o \nleftrightarrow A,\ x \leftrightarrow b,\ a \nleftrightarrow b)\,,$$

and Lemma 9.3 reads

$$\begin{aligned}\mathbb{P}(a \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow x)\,&\mathbb{P}(o \nleftrightarrow A,\ x \leftrightarrow b,\ a \nleftrightarrow b)\\ &\le \mathbb{P}(o \nleftrightarrow A,\ a \leftrightarrow b,\ x \nleftrightarrow b)\,\mathbb{P}(x \leftrightarrow b,\ o \nleftrightarrow a,\ o \leftrightarrow x)\,.\end{aligned} \tag{52}$$

Splitting $\{o \nleftrightarrow A,\ a \leftrightarrow b\}$ and $\{o \nleftrightarrow A,\ x \leftrightarrow b\}$ according as the other vertex of A reaches b,

$$\rho_a - \mathbb{P}(o \nleftrightarrow A,\ a \leftrightarrow b,\ x \nleftrightarrow b) = \mathbb{P}(o \nleftrightarrow A,\ a \leftrightarrow b,\ x \leftrightarrow b) = \rho_x - \mathbb{P}(o \nleftrightarrow A,\ x \leftrightarrow b,\ a \nleftrightarrow b)\,, \tag{53}$$

Set

$$u = \mathbb{P}(o \nleftrightarrow A, a \leftrightarrow b, x \nleftrightarrow b), \qquad v = \mathbb{P}(o \nleftrightarrow A, x \leftrightarrow b, a \nleftrightarrow b).$$

Equation (53) gives $v - u = \rho_x - \rho_a > 0$, so $0 \le u < v$. If L_a and L_x denote the left- and right-hand probabilities in (51), then (52) is $L_a v \le u L_x$. Hence $L_a \le (u/v) L_x \le L_x$, proving (51).

The case $|A| = 1$ and the strict two-vertex case prove the unique-minimizer clause.

For the positive-probability clause, assume $\mathbb{P}(o \nleftrightarrow A) > 0$ and let a realize the minimum. Again only $A = \{a, x\}$ has to be treated, and only $\rho_a = \rho_x$.

Suppose first that $\rho_a = 0$. A configuration has positive probability precisely when it contains every edge of probability one and no edge of probability zero. Let ω_1 contain exactly the probability-one edges. Choose a positive-probability configuration $\eta \in \{o \nleftrightarrow A\}$. Since $\omega_1 \subseteq \eta$ and $\{o \nleftrightarrow A\}$ is decreasing, $\omega_1 \in \{o \nleftrightarrow A\}$.

Suppose $\mathbb{P}(a \leftrightarrow b, o \nleftrightarrow b) > 0$. Choose a positive-probability configuration $\xi \in \{a \leftrightarrow b, o \nleftrightarrow b\}$ and a simple open path P from a to b in ξ. Let $\omega = \omega_1 \cup P$. This configuration has positive probability, $\omega \subseteq \xi$, and $a \leftrightarrow b$ in ω. Moreover $o \nleftrightarrow A$ in ω. Indeed, a path from o to A that uses only edges of ω_1 would contradict $\omega_1 \in \{o \nleftrightarrow A\}$. Any other such path meets P and would connect o to b in ω, hence in ξ, contradicting the choice of ξ. Thus $\rho_a > 0$, a contradiction. Hence $\{a \leftrightarrow b\}$ is contained in $\{o \leftrightarrow b\}$ up to a null event. On $\{a \leftrightarrow b\} \cap \{o \leftrightarrow b\}$, the vertices o and a are connected through b, so $\{o \leftrightarrow A\}$ holds as well; therefore $\mathbb{P}(o \leftrightarrow A, a \leftrightarrow b) \le \mathbb{P}(a \leftrightarrow b) \le \mathbb{P}(o \leftrightarrow b, o \leftrightarrow A)$.

Suppose finally that $\rho_a > 0$, and fix q with $0 < q < 1$. Enlarge the vertex set by one vertex y, set $p_{\{y,a\}} = q$ and all other edge probabilities incident to y to zero, leave every old probability unchanged, and replace A by $\{y, x\}$; write $\mathbb{P}_q$ for the enlarged measure. Condition on the state of $\{y, a\}$. If the edge $\{y, a\}$ is open, then y is joined exactly to the vertices of the cluster of a, the connections among the old vertices are unchanged, and the events $\{o \leftrightarrow \{y, x\}\}$ and $\{y \leftrightarrow b\}$ are $\{o \leftrightarrow A\}$ and $\{a \leftrightarrow b\}$. If the edge $\{y, a\}$ is closed, then y is isolated, so $\{y \leftrightarrow b\}$ is empty and $\{o \leftrightarrow \{y, x\}\}$ is $\{o \leftrightarrow x\}$. Hence

$$\mathbb{P}_q(y \leftrightarrow b,\ o \nleftrightarrow \{y, x\}) = q\rho_a, \qquad \mathbb{P}_q(x \leftrightarrow b,\ o \nleftrightarrow \{y, x\}) = q\rho_x + (1-q)\mathbb{P}(x \leftrightarrow b,\ o \nleftrightarrow x)\,,$$

$$\mathbb{P}_q(o \leftrightarrow \{y, x\},\ y \leftrightarrow b) = q\,\mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b)\,,$$

$$\mathbb{P}_q(o \leftrightarrow b,\ o \leftrightarrow \{y, x\}) = q\,\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) + (1-q)\,\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow x)\,.$$

Let

$$s_y = q\rho_a, \qquad s_x = q\rho_x + (1-q)\mathbb{P}(x \leftrightarrow b, o \nleftrightarrow x)$$

be the two quantities minimized in the enlarged graph. Since $\{o \nleftrightarrow A\} \subseteq \{o \nleftrightarrow x\}$, we have $s_x \ge \rho_x = \rho_a > q\rho_a = s_y$. Thus y is the unique minimizer in $\{y, x\}$. The unique-minimizer case just proved gives

$$q\,\mathbb{P}(o \leftrightarrow A,\ a \leftrightarrow b) \le q\,\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A) + (1-q)\,\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow x)\,. \tag{54}$$

Suppose $\delta = \mathbb{P}(o \leftrightarrow A, a \leftrightarrow b) - \mathbb{P}(o \leftrightarrow b, o \leftrightarrow A)$ is positive; then $\delta \le 1$, and (54) gives $q\delta \le (1-q)\mathbb{P}(o \leftrightarrow b, o \leftrightarrow x) \le 1 - q$ for every q with $0 < q < 1$. Taking $q = 1 - \delta/4 \in [3/4, 1)$ turns this into $(1 - \delta/4)\delta \le \delta/4$, that is $\delta \ge 3$, a contradiction. Hence $\delta \le 0$, which is (4). $\square$

Remark 9.5. It remains open whether (4) holds for the unique minimizer of ρ_x when $|A| \ge 3$. By contrast, if a minimizes $\mathbb{P}(o \leftrightarrow A, x \leftrightarrow b)$ over $x \in A$, then (1) gives (4).

We conclude by specifying the scope of the accompanying Lean verification.

10 Lean Verification

The accompanying Lean formalization checks Conjectures 1–4 and 6, Questions 5, 7, and 9, and the counterexample in Proposition 9.1. It also checks the strong pre-FKG inequality, the form of Conjecture 4 with the minimizing vertex prescribed, the stronger form of Conjecture 6, and Theorem 7.1. These declarations build on the Lean proof of Theorem 3.3, named `Percolation.Continuity.CSH.cshHolds`, in [Dev26].

The cluster-exploration inputs in Section 3 are also machine-checked. The conditional-mean identity is `Percolation.Continuity.CSH.residual_orthogonal`; the positivity and monotonicity of Ψ are `Percolation.Continuity.CSH.phiFun_nonneg` and `Percolation.Continuity.CSH.phiFun_mono`; and (10) is the form for a set of vertices `KNAll.Guarded.guardResid_sourceCluster`. The stopped estimate (12) and its monotonicity use `CovTauStarN.yS_mul_mS_le` and `CovTau.Yw_antitone_of_le`. Their set-source specialization is checked in `KN/GuardedKernel.lean` as part of the public declaration `KNAll.Guarded.guardHorizontal_nonneg`.

Two results in this paper are not part of the current Lean certificate. The formal theorem for Question 5 has the right side $\mathbb{P}(o \leftrightarrow b)$ printed by Kozma and Nitzan; the sharper right side $\mathbb{P}(o \leftrightarrow b,\ o \leftrightarrow A)$ in Theorem 6.2 is proved here. The positive two-vertex result in Theorem 9.4 is also proved only in this paper.

The principal declarations are `KNAll.genY_all` for Theorem 4.1, `KNAll.conjecture4Fixed_holds` for Theorem 5.2, and `KNAll.setSourceFixedMin_holds` for Theorem 7.1. The declarations `KNAll.conjecture1_holds`, `KNAll.conjecture2_holds`, `KNAll.conjecture2Strong_holds`, `KNAll.conjecture3_holds`, `KNAll.conjecture4_holds`, `KNAll.conjecture4_clusterProperty_holds`, and `KNAll.question7_holds` cover the consequences in Section 5. Question 5 is `KNAll.question5_holds`. Conjecture 6 and its strengthening are `KNAll.Guarded.conjecture6_holds` and `KNAll.Guarded.conjecture6Strong_holds`. Question 9 is `KNAll.question9_holds`, obtained by combining `KNAll.setSourceFixedMin_holds` with `KNAll.question9_of_setSourceFixedMin`. The counterexample to Question 8 is `KNAll.not_question8EveryMin`.

Each listed declaration compiles under Lean `4.32.0` with no `sorry` and depends only on the axioms `propext`, `Classical.choice`, and `Quot.sound`. The Mathlib commit is

`81a5d257c8e410db227a6665ed08f64fea08e997`.

References

[AD78] R. Ahlswede and D. E. Daykin, *An inequality for the weights of two families of sets, their unions and intersections*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **43** (1978), 183–185.

[BHK06] J. van den Berg, O. Häggström and J. Kahn, *Some conditional correlation inequalities for percolation and related processes*, Random Structures Algorithms **29** (2006), 417–435. Theorem 1.1 (p. 3) gives Theorem 3.1. At $q = 1$, Theorem 2.1 (p. 9) follows from Theorem 1.5 (p. 7) and Remark 1 (p. 5); these give Theorem 3.2.

[Dev26] *Theta at the critical point vanishes for Bernoulli bond percolation on the hypercubic lattice in all dimensions at least two, in Lean 4 and Mathlib*, 2026, `https://github.com/anthropic-experimental/percolation`.

[Gla24] N. Gladkov, *Percolation inequalities and decision trees*, arXiv:2408.08457v2, 2024.

[H60] T. E. Harris, *A lower bound for the critical probability in a certain percolation process*, Proc. Cambridge Philos. Soc. **56** (1960), 13–20.

[KN24] G. Kozma and S. Nitzan, *A reduction of the percolation at criticality problem to a conjectured inequality*, arXiv:2401.12397, 2024. Conjectures 1–3 with display (3) are on pp. 3 and 15,

Conjecture 4 and Question 5 on pp. 31–32, Conjecture 6 on p. 34, and Questions 7–9 with display (41) on p. 36.