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The Dirichlet problem in non-smooth domains

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Introduction

In this paper we examine the boundary behavior of solutions to the Dirichlet problem for an elliptic operator $\mathcal{L} = \sum \partial_i a_{ij} \partial_j$, with smooth coefficients a_{ij}, in domains D in $\mathbf{R}^m$, $m \geqq 3$, with non-smooth boundaries. The domains we treat are given locally, in some C^∞ coordinate system, by the graph of a continuous function ϕ, with $\nabla\phi \in L^p$, for some p, $1 \leqq p \leqq \infty$. Such domains are called L_1^p domains. (See Section 1 for the precise definitions.) Note that L_1^∞ domains are usually called Lipschitz domains. If D is an L_1^p domain, and $X \in D$, we study the "elliptic" measure ω^X, associated with $\mathcal{L}$ and D at X. Thus, for f a continuous function on ∂D, the formula $u(X) = Hf(X) = \int_{\partial D} f\, d\omega^X$ gives the Perron-Wiener-Brelot generalized solution of $\mathcal{L}u = 0$, and $u = f$ on ∂D. (See Section 1 for the precise definitions.)

In [6] and [7], B. E. J. Dahlberg proved, for $\mathcal{L} = \Delta$, and D an L_1^∞ domain, the following theorem:

THEOREM. a) *Let σ be the surface of measure of ∂D, and let $\omega = \omega^{X_0}$, $X_0 \in D$, be fixed. Then, $\sigma \ll \omega$ and $\omega \ll \sigma$.*

b) *$k \in L^2(d\sigma)$ and $\left(1/\sigma(\Delta) \int_\Delta k^2\, d\sigma\right)^{1/2} \leqq C\left(1/\sigma(\Delta) \int_\Delta k\, d\sigma\right)$, where $k = d\omega/d\sigma$.*

c) *If $f \in L^p(d\sigma)$, $p \geqq 2$, then $u(X) = Hf(X)$ converges non-tangentially to f almost everywhere $(d\sigma)$, and its non-tangential maximal function $N(u)$ satisfies $||N(u)||_{L^p(d\sigma)} \leqq C||f||_{L^p(d\sigma)}$.*

In [11], a new proof of this result, based on a simple integral identity, was announced. This proof extended part a) to (locally) star-shaped L_1^p, domains in $\mathbf{R}^m$, $p > (m - 1)$. It also showed, under appropriate exterior star-shaped assumptions, that in this case $k \in L^q(d\sigma)$, for $q = 2 - (2/p)$. The domains in question also had to be assumed to be regular for the Dirichlet problem (again, see Section 1 for the precise definitions).

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In this paper, we give detailed proofs of the results in [11], and we extend the integral identity and its consequences to arbitrary elliptic operators $\mathcal{L}$, with smooth coefficients. The coordinate-free setting has allowed us to dispense with cumbersome interior and exterior star-shaped conditions. We have also been able to drop the hypothesis of regularity.

In addition, there are two new endpoint results. For L_1^2 domains, we show that $\omega \ll \sigma$. E. M. Stein has pointed out ([16]) that functions with gradient in the Lorentz space $L^{n,1}(\mathbf{R}^n)$ are continuous and differentiable almost everywhere. Using this result we are able to show that harmonic measure and surface measure are mutually absolutely continuous in $L_1^{m-1,1}$ domains in $\mathbf{R}^m$. Notice that this is an appropriate analogue of the classical result in $\mathbf{R}^2$ that harmonic and surface measure are mutually absolutely continuous on domains with rectifiable boundary.

Finally, we treat the Dirichlet problem in L_1^p domains, $p > m - 1$, obtaining a result that is similar to the qualitative part of c) in the theorem above. We have shown that if $f \in L^r(d\sigma)$, $r \geqq 2\big((p-1)/(p-2)\big)$, then $u(X) = Hf(X)$ satisfies $\mathcal{L}u = 0$ in D, and u converges non-tangentially almost everywhere $(d\sigma)$ to f. There is also an endpoint result in $L_1^{m-1,1}$ domains. However it seems that no quantitative non-tangential maximal function estimate holds in L_1^p domains, $p < \infty$. For such estimates in more general domains than Lipschitz, see [12].

One might think, because of the results in [3], that the coefficients a_{ij} need only be bounded and measurable for our theorems to hold. However, in [2] examples are given of elliptic operators $\sum \partial_i a_{ij} \partial_j$, $a_{ij} \in L^\infty$, for which the elliptic measures $d\omega$ and $d\sigma$ are mutually singular, even on smooth domains.

In Section 1, we present the relevant definitions and notations. In Section 2 we prove the main lemmas for smooth domains, which give the necessary a priori inequalities. In Section 3, we give the estimates for elliptic measure, and in Section 4 we treat the Dirichlet problem.

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Section 1

In this section we give the definitions and set up the notations that will be used throughout the paper. We also recall the basic results on elliptic operators that will be needed in the sequel. Our main reference for elliptic operators is [14].

Definition 1.1. Let $D \subset \mathbf{R}^m$ be a bounded domain and $Q_0 \in \partial D$. We say that D is given near Q_0 by the graph of a continuous function $\phi: \mathbf{R}^{m-1} \to \mathbf{R}$, with $\phi(0) = 0$, if there exist an open set U, a C^∞ diffeomorphism $\eta: \mathbf{R}^m \to \mathbf{R}^m$, with $\eta(Q_0) = 0$, and numbers $r_0 > 0$, $s_0 > 1$ such that $|\phi(x)| < s_0 r_0$ for $|x| < r_0$, $\eta(U) = \{(x, y); |x| < r_0, |y| < s_0 r_0\}$ and $\eta(U \cap D) = \{x, y); |x| < r_0, |y| < s_0 r_0, y > \phi(x)\}$. The pair (U, η) will be called the coordinate chart corresponding to ϕ.

Definition 1.2. Let $D \subset \mathbf{R}^m$ be a bounded domain, and $Q_0 \in \partial D$. We say that D is given near Q_0 in rectilinear coordinates by the graph of a continuous function ϕ, if in the definition above we can take $\eta = \mathrm{id}$.

Definition 1.3. Let $D \subset \mathbf{R}^m$ be a bounded domain with rectifiable boundary. D is called differentiable almost everywhere if for almost every boundary point Q, D admits a tangent plane at Q. Note that if D is given near Q_0 by the graph of a function ϕ, D is differentiable almost everywhere near Q_0 if and only if ϕ is differentiable almost everywhere in $|x| < r_0$, i.e., if for almost every x, $|x| < r_0$, there exists a linear function $\Lambda(x)$ such that $|\phi(x + y) - \phi(x) - \Lambda(x)(y)| = O(|y|)$ as $|y| \to 0$.

Definition 1.4. Let $D \subset \mathbf{R}^m$ be a bounded domain. D is an $L_1^p(\mathrm{BMO}_1)$ domain if near every boundary point Q_0, D is given by the graph of a continuous function ϕ, with $\nabla\phi \in L^p(\mathrm{BMO})$. Similarly we can define $L_1^{m-1,1}$ domains as given by graphs of functions ϕ with $\nabla\phi \in L^{m-1,1}$, the Lorentz space.

Note that L_1^∞ domains are usually called Lipschitz domains.

Definition 1.5. Let $D \subset \mathbf{R}^m$ be a bounded domain. D is a C^∞ domain, with defining function ρ, if ρ is defined in a neighborhood of $\bar{D}$, ρ is C^∞, $\rho(x) < 0$ if and only if $x \in D$; $\{\rho(x) = 0\} = \partial D$, and $|\nabla\rho(x)| > 0$ if $x \in \partial D$.

We will consider operators $\mathfrak{L} = \sum_{i,j} \partial_i a_{ij}(x) \partial_j$, where $a_{ij}(x) = a_{ji}(x)$, a_{ij} are C^∞ functions in $\mathbf{R}^m$, and $\mathfrak{L}$ is uniformly elliptic, with ellipticity constant λ (i.e., $\lambda \sum_i \xi_i^2 \leqq \sum_{i,j} a_{i,j}(x)\xi_i\xi_j \leqq \lambda^{-1} \sum_i \xi_i^2$), for all $\xi = (\xi_1, \cdots, \xi_m)$ in $\mathbf{R}^m$, and all $x \in \mathbf{R}^m$. Notice that the class of these operators is invariant under diffeomorphism. We will be concerned with the Dirichlet problem for $\mathfrak{L}$ on a bounded domain D of the kind considered in Definition 1.4. We remark that these operators verify the maximum principle, and Harnack's inequality (see [14]).

We will now introduce the notion of elliptic measure for one of these operators. An extended real valued function u on D is called an $\mathfrak{L}$-supersolution if i) u is not identically $+\infty$ on D; ii) $u > -\infty$ on D; iii) u is lower semicontinuous in D; and iv) if $W \Subset D$, and h is continuous on $\bar{W}$, $\mathfrak{L}h = 0$ on W, and $u \geqq h$ on ∂W, then $u \geqq h$ on W. A function u is called an $\mathfrak{L}$-subsolution if $-u$ is an $\mathfrak{L}$-supersolution. Let f be defined on ∂D. The upper class of functions

$$\begin{aligned} U_f = \{u\colon & u \text{ is either identically } +\infty \text{ on } D, \text{ or } u \text{ is an } \mathfrak{L}\text{-supersolution in } D, \\ & \text{with } \liminf_{X\to Q} u(X) \geqq f(Q), \text{ for all } Q \in \partial D, \\ & u \text{ bounded below on } D\} \,. \end{aligned}$$

The lower class L_f is

$$\begin{aligned} L_f = \{u\colon & u \text{ is either identically } -\infty \text{ on } D, \text{ or } u \text{ is an } \mathfrak{L}\text{-subsolution in } D, \\ & \text{with } \limsup_{X\to Q} u(X) \leqq f(Q) \text{ for all } Q \in \partial D, \\ & u \text{ bounded above on } D\} \,. \end{aligned}$$

Definition 1.6. $\bar{H}f = \inf\{u, u \in U_f\}$ is the upper solution for the generalized Dirichlet problem for f. $\underline{H}f = \sup\{u, u \in L_f\}$ is the corresponding lower solution.

Definition 1.7. If $\bar{H}f = \underline{H}f = Hf$ on D, and $\mathfrak{L}(Hf) = 0$ on D, f is called a resolutive boundary function.

THEOREM 1.8 (Wiener [18]). *If f is a continuous, real-valued function on ∂D, D a bounded domain in $\mathbf{R}^m$, then f is resolutive.*

As a consequence of Wiener's theorem, it is possible to define the elliptic measure for L in D.

The unique probability Borel measure on ∂D, $\omega_{\mathfrak{L}}^X$, such that for all continuous f on ∂D, $Hf(X) = \int_{\partial D} f\,d\omega_{\mathfrak{L}}^X$, is called the elliptic measure for D, evaluated at X, associated to $\mathfrak{L}$. When $\mathfrak{L}$ is clearly understood in context, we shall simply write ω^X for $\mathfrak{L}$-elliptic measure evaluated at X.

As a consequence of Harnack's inequality, for any X_1, X_2 in D, the measures $\omega_{\mathfrak{L}}^{X_1}$ and $\omega_{\mathfrak{L}}^{X_2}$ are mutually absolutely continuous, $C \leqq (d\omega^{X_1})/(d\omega^{X_2}) \leqq 1/C$, and $L^p(d\omega^{X_1}) = L^p(d\omega^{X_2})$, with comparable norms. A function f will be said to belong to $L^1(d\omega)$ if $f \in L^1(d\omega^X)$ for some (and hence all) $X \in D$.

THEOREM 1.9 (Brelot). *Let D be a bounded domain in $\mathbf{R}^m$. A boundary function f is resolutive if and only if it is in $L^1(d\omega)$. In this case,*

$$Hf(X) = \int_{\partial D} f\,d\omega^X \quad \textit{for all} \quad X \in D\,. \qquad \text{(See [1].)}$$

Definition 1.10. Let $D \subset \mathbf{R}^m$ be a bounded domain. D is regular for the Dirichlet problem, with respect to $\mathfrak{L}$, if given any continuous function f on ∂D, one can find a function u in $C(\bar{D})$, with $\mathfrak{L}u = 0$ in D, $u|_{\partial D} = f$. Note that necessarily $u = Hf$.

We remark (see [14], IV, 28) that a domain D is regular for $\mathfrak{L}$ if and only if it is regular for the Laplacian Δ. A consequence of this is that regularity is a local property of ∂D, invariant under diffeomorphism.

Unfortunately, as we shall see later on, L_1^p domains, $p < \infty$, need not be regular. However, the set of irregular points has capacity 0, by a theorem of Kellogg ([13]), and thus has 0 surface area (see [4] for example).

We will usually be dealing with a bounded domain D, contained in a big, fixed ball B, with $D \subset B/2$, the ball with the same center as B, and half the radius. $F(X, Y)$ will denote the Green function for $\mathfrak{L}$ in B (see [14], pages 20, 21, 80 for the definition and properties of F). We will use the following estimates for F, which hold uniformly for $X, Y \in B/2$, $X \neq Y$:

(1.11) i) $$|F(X, Y)| \leqq \frac{C}{|X-Y|^{m-2}};$$

ii) $$\left|\frac{\partial}{\partial X_i} F(X, Y)\right| \leqq \frac{C}{|X-Y|^{m-1}};$$

iii) $$\left|\frac{\partial^2}{\partial X_i \partial X_j} F(X, Y)\right| \leqq \frac{C}{|X-Y|^{m}}.$$

Moreover, $F(X, Y) = F(Y, X)$, and F is unique. We note that similar properties hold for the Green function G of any smooth domain $D \subset \mathbf{R}^m$. Thus, if D is smooth, $D \subset B/2$, we have $G(X, Y) = F(X, Y) - g_X(Y)$, where, for each fixed $X \in \partial D$, $\mathfrak{L} g_X = 0$, and $g_X(Q) = F(X, Q)$, for every $Q \in \partial D$.

As an application of Stokes' theorem we obtain (see [14], p. 21, 10.4),

(1.12) $$u(X) = -\int_D G(X, Y) f(Y)\, dY + \int_{\partial D} \langle \mathfrak{N}_Q, \nabla_Q G(X, Q)\rangle \phi(Q)\, d\sigma(Q),$$

where $\mathfrak{L}u = f$ in D and $u|_{\partial D} = \phi$ and $\mathfrak{N}_Q$ is the conormal; i.e., $\mathfrak{N}_Q = A(Q)N_Q$, where N_Q is the inward pointing normal and $A(Q) = (a_{ij}(Q))$. As a consequence of this formula, we see that for a C^∞ domain D, $k^X(Q) \equiv (d\omega^X)/(d\sigma) = \langle \mathfrak{N}_Q, \nabla_Q G(X, Q)\rangle$. We will identify the vector $\mathfrak{N}_Q$ with the vector field $\mathfrak{N} = \langle \mathfrak{N}_Q, \nabla_Q \rangle$.

Section 2

In this section we establish the main lemmas needed for the proofs of our theorems.

LEMMA 2.1 (*The main identity*). *Assume* $0 \in D$, $D \subset \mathbf{R}^m$ *is a* C^∞ *domain, with defining function* ρ. *Assume* $D \subset B/2$, *and* $F(Y) = F(0, Y)$ *is the Green function of* B, *with pole at* 0. *Thus* $G(Y) = G(0, Y) = F(Y) - g(Y)$, *where* $\mathfrak{L}g = 0$, *and* $g|_{\partial D} = F|_{\partial D}$. *Let* $k(Q) = d\omega^0/d\sigma$, *and* V *be a* C^∞ *vector field, with* $V(0) = 0$. *Then,*

$$\int_{\partial D} \frac{V\rho}{\mathfrak{N}\rho} \cdot k^2 d\sigma = \int_{\partial D} (VF)k\, d\sigma - \int_D G(Y)\cdot[\mathfrak{L}, V](g)(Y)\, dY .$$

Proof. Because $\langle \mathfrak{N}_Q, N_Q\rangle = \langle A(Q)N_Q, N_Q\rangle \geqq \lambda$, we have $V(Q) = T(Q) + c(Q)\cdot\mathfrak{N}_Q$, where T is a vector field tangential to ∂D. Since G vanishes on ∂D, $TG = 0$, and so $VG = c(Q)\langle \mathfrak{N}_Q, \nabla G(Q)\rangle = c(Q)k(Q)$. Thus, $\int_{\partial D} (VG)k\, d\sigma = \int_{\partial D} c(Q)k^2(Q)\, d\sigma$. Now $V\rho = T\rho + c(Q)\mathfrak{N}\rho$, and $T\rho = 0$. Note that $\mathfrak{N}\rho \neq 0$. In fact,

$$\mathfrak{N}\rho = \langle A(Q)N_Q, \nabla\rho(Q)\rangle = \left\langle A(Q)\frac{\nabla\rho(Q)}{|\nabla\rho(Q)|}, \nabla\rho(Q)\right\rangle \cong |\nabla\rho(Q)| .$$

Thus $c(Q) = V\rho/\mathfrak{N}\rho$. Moreover, $VG = VF - Vg$, and so,

$$\int_{\partial D} \frac{V\rho}{\mathfrak{N}\rho} k^2 d\sigma = \int_{\partial D} (VF)k\, d\tau - \int_{\partial D} (Vg)k\, d\sigma .$$

Since $\mathfrak{L}(Vg)(0) = 0$, (1.12) implies that

$$\int_{\partial D} (Vg)k\, d\sigma = \int_D G(Y)\mathfrak{L}(Vg)(Y)\, dY .$$

But, $\mathfrak{L}g = 0$, and so our identity follows.

We remark that when $\mathfrak{L} = \Delta$, and $Vf = X\cdot\nabla f$, this identity coincides with the one given in [11].

LEMMA 2.2. *Let $D \subset \mathbf{R}^m$ be a bounded C^∞ domain, $0 \in D$. Let (U, η), Q_0 and ϕ be as in Definition* 1.1. *Assume $W \Subset U$, and let $\Delta = W \cap \partial D$. Then*

$$\int_\Delta \frac{k^2(Q)}{(1 + |\nabla\phi|^2)^{1/2}}\, d\sigma \leqq C ,$$

where C depends only on dist$(0, \partial D)$, dist$(0, \bar{U})$, W, U, η, *bounds for the a_{ij} and their derivatives on a ball B such that $D \subset B/2$, and on the ellipticity constant λ.*

Proof. Replacing $\mathfrak{L}$ by another operator of the same kind (again denoted by $\mathfrak{L}$) we can assume that D is given near Q_0 in rectilinear coordinates by the graph of ϕ, and that $W \Subset U$ (see Definition 1.2). The origin in the old coordinates goes to a point P_0 in the rectilinear coordinates.

Pick $\theta \in C^\infty$, $\theta \equiv 1$ on W, $\theta \equiv 0$ outside of U, and define $Vf(z) = \theta(z)(\partial/\partial y)f(z)$, where $z = (x, y)$. On supp θ, we can take $\rho(z) = y - \phi(x)$ as a defining function of D. We recall that $\mathfrak{N}\rho \cong |\nabla\rho|$, and thus, for $z \in$ supp θ, $\mathfrak{N}\rho \cong (1 + |\nabla\phi|^2)^{1/2}$. Also, $V\rho = \theta(z)$, and thus, Lemma 2.1 implies that

$$\int_\Delta \frac{k^2}{(1 + |\nabla\phi|^2)^{1/2}}\, d\sigma \leqq c\left|\int_{\partial D} (VF)k\, d\sigma\right| + c\left|\int_D G(Y)[\mathfrak{L}, V](g)(Y)\, dY\right| .$$

On the other hand, (1.11) ii) shows that $|\nabla F(X)| \leqq C/|X - P_0|^{m-1}$, and thus, since $P_0 \notin U$, $\operatorname{supp} \theta \subset U$, $|VF| \leqq C$. Hence, the first term has the correct bound. From now on, integrals of the form $\int_D f$ and $\int_{\tilde{W} \cap D} f$ will represent integration with respect to volume. The volume element will be omitted unless a surface integral appears in the same formula.

Now, let

$$S = [\mathfrak{L}, V] = \sum_{i,j} b_{ij}(x)\partial_i\partial_j + \sum_i c_i(x)\partial_i = \sum_{i,j} \partial_i\big(d_{ij}(x)\big)\partial_j + \sum_i l_i(x)\partial_i ,$$

where the d_{ij}, the l_i vanish outside of $\operatorname{supp} \theta \subset U$ and have bounds with the right dependence. Let $\tilde{W} = \operatorname{supp} \theta$. Since G vanishes on ∂D, using the divergence theorem and the remarks above, we see that

$$\left|\int_D G[\mathfrak{L}, V](g)\right| \leqq C\left\{\int_{\tilde{W} \cap D} |\langle \nabla G, \nabla g\rangle| + \int_{\tilde{W} \cap D} G|\nabla g|\right\} = I + II ;$$

$$I \leqq \int_{\tilde{W} \cap D} |\nabla F|\,|\nabla g| + \int_{\tilde{W} \cap D} |\nabla g|^2$$

$$\leqq \left(\int_{\tilde{W} \cap D} |\nabla F|^2\right)^{1/2}\left(\int_{\tilde{W} \cap D} |\nabla g|^2\right)^{1/2} + \int_{\tilde{W} \cap D} |\nabla g|^2 .$$

In $\tilde{W} \cap D$, $|\nabla F| \leqq C$, and $|\tilde{W} \cap D| \leqq |B|$. Thus, all we have to estimate is the term involving ∇g. Let $d = \min\{\operatorname{dist}(P_0, \partial D), \operatorname{dist}(P_0, \bar{U})\}$. Let $\chi \in C^\infty(\mathbf{R}^m)$, $\chi \equiv 1$ outside $B(P_0, d)$, $\chi \equiv 0$ in $B(P_0, d/2)$. Let w solve

$$(*) \qquad \begin{cases} \mathfrak{L}w = \mathfrak{L}\chi F \\ w|_{\partial D} = 0 . \end{cases}$$

Then, $g = \chi F - w$. As before, $\int_{\tilde{W} \cap D} |\nabla(\chi F)|^2 \leqq C$. Thus, we want to bound $\int_{\tilde{W} \cap D} |\nabla w|^2 \leqq \int_D |\nabla w|^2$. But

$$\int_D |\nabla w|^2 \cong \int_D \langle A\nabla w, \nabla w\rangle = -\int_D (\mathfrak{L}w)\cdot w = -\int_D \mathfrak{L}(\chi F)\cdot w$$

by (*) and the divergence theorem. Moreover,

$$\left|\int_D \mathfrak{L}(\chi F)\cdot w\right| \leqq \left(\int_D |\mathfrak{L}(\chi F)|^2\right)^{1/2}\left(\int_D |w|^2\right)^{1/2} ,$$

and $\|\mathfrak{L}(\chi F)\|_\infty \leqq C$. The maximum principle applied to g shows that $|w| = |g - \chi F| \leqq C$, and thus I is controlled. II can be estimated in the same fashion, and thus the lemma is established.

LEMMA 2.3. *Let $D \subset \mathbf{R}^m$ be a bounded C^∞ domain. Let (U, η), Q_0 and ϕ be as in Definition* 1.1. *Assume that $2r < r_0$, and let $\Delta_r = \eta^{-1}\{(x, \phi(x))$; $|x| < r\}$, $Z_r = \eta^{-1}(0, 2s_0r)$. If $k_r = (d\omega^{Z_r})/(d\sigma)$, then $\int_{\Delta_r} k_r^2\, d\sigma \leqq c/r^{m-1}$, where*

C depends only $\|\nabla\phi\|_\infty$, r_0, η, bounds for the a_{ij} and their derivatives on a ball B with $D \subset B/2$, and on the ellipticity constant λ.

Proof. The proof is essentially the same as the one of Lemma 2.2, if we replace 0 by Z_r, and keep careful track of the dependence on r of the constants appearing. As usual, we may assume that $\eta = \mathrm{id}$. Thus, $\Delta_r = \{(x, \phi(x)); |x| < r\}$. Let $\tilde{W} = \{(x, y); |x| < (3/2)r, |y| < (3/2)s_0 r\}$, $W = \{(x, y); |x| < r, |y| < s_0 r\}$, and pick $\theta \equiv 1$ in W, 0 outside of $\tilde{W}$. We define V as in Lemma 2.2, and we thus get

$$\left|\int_{\Delta_r} k_r^2\, d\sigma\right| \leqq C\left|\int_{\partial D}(VF)k_r\, d\sigma\right| + C\left|\int_D G(Y)[\mathfrak{L}, V](g)(Y)\, dY\right| ,$$

where C has the correct dependence. We now note that

$$(\dagger) \qquad \operatorname{dist}(Z_r, \tilde{W}) \cong r , \qquad \operatorname{dist}(Z_r, W) \cong r , \qquad \operatorname{dist}(Z_r, \partial D) \cong r ,$$

where the constants depend only on $\|\nabla\phi\|_\infty$. Also, $|\nabla F(X)| \leqq C/|X - Z_r|^{m-1}$, and so, on $\operatorname{supp} VF$, $|VF| \leqq C/r^{m-1}$. Thus, we need only show $\left|\int_D G(Sg)\right| \leqq C/r^{m-1}$, where $S = [\mathfrak{L}, V]$. Since θ is supported in a cylinder of size r, we now have $S = (1/r)\sum_{i,j}\partial_i(d_{ij}(x))\partial_j + (1/r^2)\sum_i l_i(x)\partial_i$, where d_{ij} and l_i vanish outside of $\tilde{W}$, and have bounds independent of r. Thus,

$$\left|\int_D G[Sg]\right| \leqq C\left\{\frac{1}{r}\int_{\tilde{W}\cap D}|\langle\nabla G, \nabla g\rangle| + \frac{1}{r^2}\int_{\tilde{W}\cap D} G|\nabla g|\right\} = I + II$$

where C has the right dependence. We will estimate I, as before, the estimate for II being analogous.

$$I \leqq \frac{1}{r}\left(\int_{\tilde{W}\cap D}|\nabla F|^2\right)^{1/2}\left(\int_{\tilde{W}\cap D}|\nabla g|^2\right)^{1/2} + \frac{1}{r}\int_{\tilde{W}\cap D}|\nabla g|^2 .$$

By ($\dagger$), in $\tilde{W}\cap D$, $|\nabla F| \leqq C/r^{m-1}$. All we have to show then is that $\int_{\tilde{W}\cap D}|\nabla g|^2 \leqq C/(r^{m-2})$. Let $d = \min\{\operatorname{dist}(Z_r, \partial D), \operatorname{dist}(Z_r, \tilde{W})\}$. By ($\dagger$), $d \cong r$. Choose χ as in Lemma 2.2, and again, let w solve (*). As in Lemma 2.2, $g = \chi F - w$. Since $\nabla(\chi F) = \nabla F$ on $\tilde{W}\cap D$, we see that $\int_{\tilde{W}\cap D}|\nabla(\chi F)|^2 \leqq C/r^{(m-2)}$. Next, we have to estimate

$$\int_D |\nabla w|^2 \cong \int_D \langle A\nabla w, \nabla w\rangle = \left|\int_D \mathfrak{L}(\chi F)\cdot w\right| .$$

By Hölder's inequality, this last integral is less than or equal to

$$\left(\int_D |\mathfrak{L}(\chi F)|^{4/3}\right)^{3/4}\cdot\left(\int_D |w|^4\right)^{1/4} .$$

For the first factor, we note that $\operatorname{supp}\mathfrak{L}(\chi F) \subset \mathbf{R}^m\backslash B(Z_r, d/2)$, and so, $|\mathfrak{L}(\chi F)| \leqq C/|X - Z_r|^m$ on its support. Thus, the first factor is less than

$\left(\int_r^\infty t^{-m4/3} t^{m-1}\,dt\right)^{3/4} = cr^{-m/4}$. For the second factor, the maximum principle applied to g shows that $|w| = |g - \chi F| \leqq 2|F| \leqq C/|X - Z_r|^{m-2}$. Also, again by the maximum principle, $|w| \leqq C/r^{m-2}$, and thus in polar coordinates,

$$\left(\int_D |w|^4\right)^{1/4} \leqq C\left\{\int_r^\infty t^{-4(m-2)} t^{m-1}\,dt + \int_0^r \frac{t^{m-1}}{r^{4(m-2)}}\,dt\right\}^{1/4} \leqq \frac{C}{r^{(3m/4)-2}}\,.$$

The integrals are convergent because $m \geqq 3$. Putting together these last two estimates, the lemma follows.

Finally, we need the following real variable lemma:

LEMMA 2.4. *Assume that ϕ is continuous, negative, and differentiable at almost every point x of a ball $\bar{B}$ centered at the origin in $\mathbf{R}^{m-1}$. Then, given $\varepsilon > 0$, there exists a closed set $F \subset B$, with $|B \backslash F| < \varepsilon$, and a Lipschitz function ψ on $\mathbf{R}^{m-1}$, with $\phi = \psi$ on F, $\phi \leqq \psi < 0$ in B.*

Proof. Because ϕ is continuous, it is uniformly bounded in absolute value by M on B. Also, as ϕ is differentiable, almost everywhere in B, if $\Lambda(x)$ verifies $|\phi(x+y) - \phi(x) - \Lambda(x)(y)| = o(|y|)$ for almost everywhere x in B, the function $x \to \Lambda(x)$ is measurable, and finite almost everywhere. Thus, given $\varepsilon > 0$, there exists a constant M_ε and a closed set $F_1 \subset B$, with $|B \backslash F_1| \leqq \varepsilon/2$, and, for all x in F_1, all y with $|y| \leqq 1/M_\varepsilon$, $|\phi(x + y) - \phi(x)| \leqq M_\varepsilon |y|$ (*). By Theorem 2, page 248 of [15], we can find a closed set $F \subset F_1$, with $|F_1 \backslash F| \leqq \varepsilon/2$, and a Lipschitz function g on $\mathbf{R}^{m-1}$, with $\phi = g + b$, and $b \equiv 0$ on F. We now claim that there exists a constant C_ε such that, for all $x \in B$, $|b(x)| \leqq C_\varepsilon \delta(x)$, where $\delta(x) = \operatorname{dist}(x, F)$. In fact if $\delta(x_0) > 1/M_\varepsilon$, our estimate follows because ϕ and g are bounded and $b(x_0) = \phi(x_0) - g(x_0)$. Assume $\delta(x_0) \leqq 1/M_\varepsilon$. Then, there exists an x in F, with $|x - x_0| = \delta(x_0) \leqq 1/M_\varepsilon$. Let $y = x_0 - x$. Since $\phi(x) = g(x)$, we have $b(x_0) = \phi(x_0) - \phi(x) - g(x_0) + g(x)$. Thus, (*) and the Lipschitz character of g imply our claim. Now let $\tilde{\psi} = g + C_\varepsilon \delta(x)$, $\psi = \min(\tilde{\psi}, 0)$. ψ and F verify the lemma.

Section 3

In this section we establish our estimates for elliptic measure.

THEOREM 3.1. *Let D be a bounded L_1^p domain in $\mathbf{R}^m$. Let ω be the elliptic measure for $\mathcal{L}$, with respect to $X_0 \in D$.*

a) *If $2 \leqq p \leqq \infty$, then $\omega \ll \sigma$, and $k \in L^q(d\sigma)$, $q = 2 - 2/p$, where $k = d\omega/d\sigma$.*

b) *If $p = \infty$, then, for every surface ball $\Delta = B \cap \partial D$, B an m-dimensional ball centered at $Q_0 \in \partial D$,*

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^2\,d\sigma\right)^{1/2} \leqq C\frac{1}{\sigma(\Delta)}\int_\Delta k\,d\sigma\ .$$

c) *If D is differentiable almost everywhere, and $p \geqq 1$, then $\sigma \ll \omega$.*

Proof. a) By the definition of an L_1^p domain, to establish part a) it is enough to show that if (U, η) is a coordinate chart corresponding to the continuous L_1^p function ϕ, and $W \Subset U$, if $E \subset W \cap \partial D$, E closed and $\sigma(E) = 0$, then $\omega(E) = 0$, and if $k = d\omega/d\sigma$, $\int_{W\cap\partial D} k^q\,d\sigma < +\infty$. Replacing $\mathfrak{L}$ by another operator of the same class (again denoted by $\mathfrak{L}$), we can assume that D is given near $0 \in \partial D$, in rectilinear coordinates, by the graph of a continuous function ϕ, with $\nabla\phi \in L^p$, and $0 \in W \Subset U$ (see Definition 1.2).

Consider first the case when $p > 2$ and we can find a domain $\Omega \supset D$, with $U \cap \partial\Omega \supset U \cap \partial D$, and such that there exist C^∞ domains Ω_i, which increase to Ω, and such that in some C^∞ coordinate system,

$$\partial\Omega_i \cap \{(x, y)\colon |x| < r_0,\ |y| < s_0 r_0\} = \{(x, \phi_i(x));\ |x| < r_0\}\ ,$$

where $\phi_i \in C^\infty$, $\phi_i \to \phi$ uniformly, $\nabla\phi_i \to \nabla\phi$ in L^p when $p < \infty$, and almost everywhere when $p = \infty$, and $||\nabla\phi_i||_p \leqq C||\nabla\phi||_p$. Let $q = 2 - 2/p$, and let f be any nonnegative, continuous function, with $\operatorname{supp} f \subset \Delta = W \cap \partial D$, $\int_{\partial D} f^{q'}\,d\sigma = 1$. Since continuous functions are resolutive and $\Omega \supset D$,

$$\omega(f) = \int_{\partial D} f\,d\omega_D^{X_0} \leqq \int_\Delta f\,d\omega_\Omega^{X_0}\ .$$

Let G be a continuous function on $\mathbf{R}^m$, $\operatorname{supp} G \subset W$, such that $G|_{\partial\Omega} = f$. Because $H_\Omega f = \lim_{i\to\infty} H_{\Omega_i} G$ (see [1], page 100), we see that $\int_\Delta f\,d\omega_\Omega^{X_0} = \lim_{i\to\infty}\int_{\Delta_i} G\,d\omega_{\Omega_i}^{X_0}$, where $\Delta_i = W \cap \partial\Omega_i$. Therefore, to establish a) it is enough to show that

$$\left|\lim_{i\to\infty}\int_{\Delta_i} G\,d\omega_{\Omega_i}^{X_0}\right| \leqq C\ .$$

Let $k_i = d\omega_{\Omega_i}^{X_0}/d\sigma_i$, where $\sigma_i =$ surface measure of $\partial\Omega_i$. By Hölder's inequality,

$$\left|\int_{\Delta_i} G\,d\omega_{\Omega_i}^{X_0}\right| \leqq \left(\int_{\Delta_i} G^{q'}\,d\sigma_i\right)^{1/q'}\left(\int k_i^q\,d\sigma_i\right)^{1/q}\ .$$

Obviously, $\lim_{i\to\infty}\int_{\Delta_i} G^{q'}\,d\sigma_i = \int_\Delta f^{q'}\,d\sigma = 1$. On the other hand, by Lemma 2.2 applied to Ω_i (if $1/s + q/2 = 1$), we see that

$$\int_{\Delta_i} k_i^q(1 + |\nabla\phi_i|^2)^{-q/4}(1 + |\nabla\phi_i|^2)^{q/4}\,d\sigma_i$$
$$\leqq \left(\int_{\Delta_i} k_i^2(1 + |\nabla\phi_i|^2)^{-1/2}\,d\sigma_i\right)^{q/2}\left(\int_{\Delta_i}(1 + |\nabla\phi_i|^2)^{qs/4}\,d\sigma_i\right)^{1/2} \leqq C\ ,$$

since $(q/4)s + 1/2 = p/2$. Thus, part a) follows in this case. The general case for $p > 2$ follows from this case, because if ${}^c\Omega = \{(x, y): |x| < r_0, |y| < s_0 r_0, y < \phi(x)\}$, Ω satisfies all the required properties, except that it is unbounded, and the Ω_i are also unbounded. However, we can return to the bounded case by picking an $R \in \Omega\backslash \bar{W}$ and inverting by the Kelvin transform $K(X) = (X - R)/|X - R|^2$. The change of variables given by K transforms the operator $\mathfrak{L}$ to another one of the same kind, and hence our result follows from the previous case.

For $p = 2$, we need only check absolute continuity. Suppose that $E \subset \partial D \cap W$, $\sigma(E) = 0$, and E is closed. For any $\varepsilon > 0$ there is a continuous function f, $0 \leqq f \leqq 1$, supported in $W \cap \partial D$, such that $f \equiv 1$ on E and $\sigma(\operatorname{supp} f) \leqq \varepsilon$.

The following formal argument can be justified by the limiting procedure above.

$$\omega(E) \leqq \int f d\omega = \int f \cdot k(1 + |\nabla\phi|^2)^{1/2} dx \leqq \left(\int f \cdot k^2 dx\right)^{1/2}\left(\int f(1 + |\nabla\phi|^2) dx\right)^{1/2}$$
$$\leqq C\left(\int_{\operatorname{supp}(f)} (1 + |\nabla\phi|^2) dx\right)^{1/2} = o(1) \quad \text{as} \quad \varepsilon \longrightarrow 0 .$$

Part b) follows in a very similar manner. We can assume $\Delta = \Delta_r = \{(x, \phi(x)), |x| < r\}$ with $2r < r_0$. Let $Z_r = (0, 2s_0 r)$, and $W = \{(x, y); |x| < (3/2)r, |y| < (3/2)s_0 r\}$. The required estimate follows immediately from the two estimates:

(1) $k(Q) \leqq Ck_r(Q)\omega(\Delta_r)$ almost everywhere $Q \in \Delta_r$, where $k_r = \dfrac{d\omega^{Z_r}}{d\sigma}$.

(2) $$\int_\Delta k_r^2 \, d\sigma \leqq \frac{C}{\sigma(\Delta_r)} .$$

(1) follows as in Hunt-Wheeden [10] (see [3] for the analogue of (1) for elliptic operators $\mathfrak{L} = \sum_{i,j} \partial_i a_{ij} \partial_j$, a_{ij} merely bounded and measurable).

(2) follows if we argue exactly as in part a), but replace Lemma 2.2 by Lemma 2.3.

(c) For $p = \infty$, $\sigma \ll \omega$ follows from b) by an argument due to Gehring ([9]). Assume that D is differentiable almost everywhere and L_1^p, for some p, $1 \leqq p < \infty$. Again, it is enough to show that if W is as in part a), and $E \subset W \cap \partial D$, E closed, $\omega(E) = 0$, then $\sigma(E) = 0$. Since $\sigma(E) = \int_{\tilde{E}} (1 + |\nabla\phi|^2)^{1/2} dx$, where $\tilde{E} = \{x \in \mathbf{R}^{m-1}; (x, \phi(x)) \in E\}$, it is enough to show that $|\tilde{E}| = 0$. Let $\varepsilon > 0$ be given, and by Lemma 2.4, choose a closed set $F \subset \{x \in \mathbf{R}^{m-1}; |x| < r_0\} = B_{r_0}$, with $|B_{r_0}\backslash F| < \varepsilon$, and an L_1^∞ function ψ, with $\phi \leqq \psi < s_0 r_0$, and $\phi = \psi$ on F. Let $\Omega = \{(x, y); |x| < r_0, |y| < s_0 r_0, y > \psi(x)\}$. Then, Ω is a

Lipschitz domain, and $\Omega \subset D$. Let $E_1 = \{(x, \phi(x)); x \in E \cap F\}$. Then, $E_1 \subset E$, E_1 is closed, and $E_1 \subset \partial\Omega \cap \partial D$. Since E_1 is closed, χ_{E_1} is resolutive, and so, for $X \in \Omega$, $\omega_\Omega^X(E_1) \leqq \omega_D^X(E_1) = 0$. Thus, by the case $p = \infty$ of part c), $\sigma_\Omega(E_1) = 0$, and hence, $|\tilde{E} \cap F| = 0$. But then, $|\tilde{E}| \leqq |B_{r_0} \backslash F| \leqq \varepsilon$, and since $\varepsilon > 0$ is arbitrary, c) follows.

COROLLARY. *If D is an $L_1^{m-1,1}$ domain, then harmonic measure and surface measure are mutually absolutely continuous.*

This follows from Theorem 3.1 and the following theorem of E. M. Stein ([16]). Let $n = m - 1$.

THEOREM. *If ϕ is defined on $\mathbf{R}^n$, and $\nabla\phi \in L^{n,1}$, then ϕ is continuous and differentiable almost everywhere.*

Notice that this implies that if $\nabla\phi \in L^p$, $p > n$, then ϕ is continuous and differentiable almost everywhere (see, for example, [15], Theorem 1 page 242).

Section 4

In this section, we turn to the boundary behavior of solutions of $\mathcal{L}u = 0$ in bounded L_1^p domains, which are differentiable almost everywhere. We also treat the Dirichlet problem, with boundary data on $L^r(d\sigma)$, r sufficiently large, for such domains.

We begin with several definitions. By a cone, we mean a circular, open, possibly truncated cone, which is convex. Let D be a domain, $Q \in \partial D$, a cone Γ, with vertex at Q, is called a non-tangentical cone, if there is a cone Γ', with $\bar{\Gamma} - \{Q\} \subset \Gamma' \subset D$. A function u is said to have a non-tangential limit L at $Q \in \partial D$ if $u(X) \to L$ as $X \to Q$, for $x \in \Gamma$, for all non-tangential cones Γ with vertex at Q. A function u is said to be non-tangentially bounded at $Q \in \partial D$ if there exists a constant M and a non-tangential cone Γ, with vertex at Q such that $|u(X)| \leqq M$, for all $x \in \Gamma$. Analogously we define non-tangentially bounded from above (or below) at Q_0.

We observe that if $\tilde{D} \subset D$, $Q \in \partial D \cap \partial\tilde{D}$, and both D and $\tilde{D}$ are differentiable at Q, with a common tangent plane, then, if Γ is a cone with vertex at Q, and non-tangential with respect to D, it is non-tangential with respect to $\tilde{D}$, provided the diameter of Γ is sufficiently small.

Assume now that D is a bounded Lipschitz domain, and to each $Q \in \partial D$ there is associated a cone $\Gamma(Q)$, with vertex Q. $\{\Gamma(Q)\}$ is a regular family of cones, if we can partition ∂D into finitely many subsets F_j, such that to each F_j, there are cones Γ_1, Γ_2, Γ_3, with vertex at 0, such that for $Q \in F_j$, we have

$$\Gamma_1 + Q \subset \Gamma(Q) \subset \gamma + Q \subset \Gamma_3 + Q \subset D\,, \quad \text{where} \quad \gamma = \bar{\Gamma}_2 - \{0\}\,.$$

For any function u on D, and $\{\Gamma(Q)\}$ a regular family of cones, we define the non-tangential maximal function of u, $N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|$.

THEOREM 4.1. *Assume that D is a bounded Lipschitz domain in $\mathbf{R}^m$.*

a) *If $f \in L^p(d\sigma)$, $p \geqq 2$, there exists a function u in D, satisfying $\mathfrak{L}u = 0$, with non-tangential maximal function $N(u)$ in $L^p(d\sigma)$, and such that u converges non-tangentially to f at almost every $(d\sigma)$ boundary point Q.*

b) *Assume $E \subset \partial D$, and u is a function defined on D, such that $\mathfrak{L}u = 0$, and u is non-tangentially bounded from below at every $Q \in E$. Then, u has a finite non-tangential limit at almost every $(d\sigma)$ point of E.*

Proof. a) By Theorem 3.1 a), $f \in L^1(d\omega)$. Let $u = \int_{\partial D} f d\omega^X = Hf(X)$. Also, let $Mf(Q) = \sup_{Q \in \Delta} \left(1/\omega(\Delta)\right) \int_\Delta |f|\, d\omega$, where Δ denotes a surface ball. Arguing as in Hunt-Wheeden ([10]), we see that $N(u)(Q) \leqq CMf(Q)$. (See [3] for the proof of this estimate in the case when $\mathfrak{L} = \sum_{i,j} \partial_i a_{ij} \partial_j$, a_{ij} merely bounded and measurable.) Theorem 3.1 b) shows that $d\sigma \in A_2(d\omega)$ ([5]). The weighted norm inequality of Muckenhoupt (see [5]) shows that $\left(\int_{\partial D} (Mf)^p\, d\sigma\right)^{1/p} \leqq C\left(\int_{\partial D} |f|^p\, d\sigma\right)^{1/p}$, $p \geqq 2$. Thus, $N(u) \in L^p(d\sigma)$. The non-tangential convergence is deduced from this fact in a standard manner, if we remember that the result holds when $f \in C(\partial D)$.

b) As is well known (see [8] and [10]), b) can be deduced from a) by standard arguments. The main lemma needed is

LEMMA 4.2. *Let D be an L_1^1 domain that is differentiable almost everywhere. For any harmonic function u non-tangentially bounded from below on a subset $E \subset \partial D$, there exist star-shaped Lipschitz domains $\{D_i\}_{i=1}^\infty$ such that u is bounded on D_i. D_i and D have the same tangent plane at almost every $(d\sigma)$ on $E \cap \partial D_i$ and $\sigma(E \backslash \bigcup_i \partial D_i) = 0$.*

Lemma 4.2 follows from a combination of arguments in [8] and [10]. This lemma reduces part b) to the case where u is bounded and D is star-shaped. A weak-star compactness argument shows that $u = Hf$, for some $f \in L^\infty$, and thus the result follows from a).

Arguing as in part b), one can also obtain (see [8]):

THEOREM 4.3. *Assume D is a bounded domain in $\mathbf{R}^m$ and $E \subset \partial D$ is such that to each $Q \in E$ there is a cone $\gamma(Q)$ with vertex at Q, such that $\gamma(Q) \subset {}^cD$. Let u verify $\mathfrak{L}u = 0$ in D, and suppose that u is non-tangentially bounded from below at every point $Q \in E$. Then, u has a finite non-tangential limit* *at all points $Q \in E$, except for a set of vanishing $(m-1)$-dimensional Hausdorff measure.*

Our last result is the following:

THEOREM 4.4. *Let D be a bounded L_1^p domain, $p \geqq 1$, differentiable almost everywhere.*

a) *If $\mathfrak{L}u = 0$, and u is non-tangentially bounded from below on $E \subset \partial D$, then u has a finite non-tangential limit at almost every $(d\sigma)$ point in E.*

b) *If $f \in L^1(d\omega)$, then $u = Hf$ satisfies $\mathfrak{L}u = 0$, and u converges non-tangentially to f for almost every $(d\sigma)$ boundary point of D.*

In particular, if D is an L_1^p domain, differentiable almost everywhere, with $p \geqq 2$, and $f \in L^{q'}(d\sigma)$, $1/q' + 1/q = 1$, $q = 2 - 2/p$, then $f \in L^1(d\omega)$, and $u = Hf$ satisfies $\mathfrak{L}u = 0$, and u converges non-tangentially almost everywhere $d\sigma$ to f.

Proof. a) is a particular case of Theorem 4.3. For b), we can assume $f \geqq 0$, and thus $u \geqq 0$. By part a), u has a finite non-tangential limit at almost every $(d\sigma)$ point in ∂D. Our task is to show that this limit equals f almost everywhere $(d\sigma)$.

Consider the domains D_i from Lemma 4.2 for $E = \partial D$. We have only to show that for almost every $(d\sigma)$ point in $\partial D_i \cap \partial D$, u converges non-tangentially to f. Let $E_i = \partial D_i \cap \partial D$, and let v be the harmonic extension to D_i of $f \cdot \chi_{E_i}$. Let M_i be a bound for u in D_i, and define w on D_i to be the harmonic extension of g, where g equals f on E_i, and M_i on $\partial D_i \backslash E_i$. Note that since $f \in L^1(d\omega)$, $f \cdot \chi_{E_i}$ is in $L^1(d\omega_{D_i})$ and so is g. Also, the fact that f is resolutive easily implies that $v \leqq u \leqq w$ on D_i. Moreover, the estimate $N(w) \leqq M(g)$ on D_i implies that for $g \in L^1(d\omega_{D_i})$, $w = Hg$ converges non-tangentially almost everywhere $(d\sigma_i)$ to g. By the analogous statement for v and $f \cdot \chi_{E_i}$, we obtain the desired result for u.

Even though Theorem 4.4 is valid for L_1^p domains, $p > m - 1$, or $L_1^{m-1,1}$ domains, these domains need not be regular.

PROPOSITION 4.5. a) *If $m = 3$, then every Λ_β domain is regular for $\beta > 0$.*

b) *If $m \geqq 4$, then a domain given locally by the graph of functions in* $\mathrm{Lip}\left(h(\log(1/h))^{1/m-3}\right)$ *is regular.*

c) *The exponent $1/(m-3)$ in b) is sharp. Regularity fails for the domain below the graph of $\phi(x) = |x|(\log 1/|x|)^\alpha$, $\alpha > 1/(m-3)$.*

Recall that if $p > 2$, $L_1^p(\mathbf{R}^2) \subset \Lambda_\beta(\mathbf{R}^2)$ for some $\beta > 0$ ([15]). It follows from 4.5 a) that when $m = 3$, every L_1^p domain is regular for $p > 2$. However, $L_1^{2,1}$ domains in $\mathbf{R}^3$ need not be regular. For example, the spine of Lebesgue, that is the domain below the graph of $\phi(x) = (\log 1/|x|)^{-1}$ is not regular. Moreover, the domain in 4.5 c) is an L_1^p domain in $\mathbf{R}^m$, $m \geqq 4$, for all $p < \infty$, for which regularity fails.

The proposition is an application of Wiener's test ([17]), which we will now state. Denote $N(x) = |x|^{2-m}$, $m \geqq 3$. The capacity of a compact set K in $\mathbf{R}^m$ is given by

$$\mathcal{C}(K) = \sup\{\mu(K) \mid \mu \text{ is a positive Borel measure and } \mu * N(x) \leqq 1 \text{ for } x \in K\} . \tag{4.6}$$

Alternatively (see [4]),

$$\mathcal{C}(K) = \inf\{\mu(K) \mid \mu \text{ is a positive Borel measure and } \mu * N(x) \geqq 1 \text{ for all } x \in K\} . \tag{4.7}$$

A point $Q \in \partial D$ is regular if Hf is continuous at Q for all continuous functions f on ∂D. Let $A_k = (\bar{B}(Q, 2^{-k}) \backslash B(Q, 2^{-k-1})) \cap {}^cD$. Wiener's test says that Q is a regular point if and only if $\sum_{k=1}^{\infty} \mathcal{C}(A_k) 2^{k(m-2)}$ diverges.

Let $R = \{(x, y) \mid x \in \mathbf{R}^{m-1}, y \in \mathbf{R}, |x| \leqq r, |y| \leqq Mr\}$. If we use 4.6 and 4.7 and the test measure $\mu = c\chi_R$, it is easy to see that for $r < 1$, $M > 1$,

$$\mathcal{C}(R) \cong \begin{cases} Mr^{m-2} , & m \geqq 4 . \\ \dfrac{M}{\log M} r , & m = 3 . \end{cases} \tag{4.8}$$

Proposition 4.5 follows from a routine calculation using the Wiener test and 4.8.

Finally, let us remark that one can recapture regularity for $m \geqq 4$ in Zygmund domains, that is, domains given locally as the graphs of functions in Λ_1, the Zygmund class. In particular, since $\mathrm{BMO}_1 \subset \Lambda_1$, BMO_1 domains are regular. In this context one can also estimate the non-tangential maximal function (see [12]).

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