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Discrete Solutions of Dirichlet Problems, Finite Volumes, and Circle Packings*

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Abstract. Convergence results for discrete solutions of Dirichlet problems for Poisson equations are obtained, where discrete solutions are constructed for triangular grids using finite volumes with sides perpendicular to, but not necessarily bisecting, corresponding edges in underlying triangulations. A method, based on properties of circle packings, is described for generating triangular meshes and associated volumes. Also, the approximation of exit probabilities of the Brownian motion by exit probabilities of random walks on circle packings is discussed.

1. Introduction

This paper originated from our studies of discrete harmonic functions given by circle packings. Such functions were introduced in [Du3] (see also [Du1]) to deal with the type problem for random walks on infinite planar graphs and the type problem for circle packings. Here we are interested in properties of these mappings, in particular, in their connections with classical harmonic functions and approximation issues.

Through most of this paper we actually work with a larger family of maps than the class of discrete harmonic functions given by circle packings. This family can briefly be described as consisting of piecewise affine functions, defined for triangulations in the plane, that are solutions of systems of linear equations derived from classical Poisson equations using finite volumes and integration. Finite volume techniques in solving

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differential equations have been studied in the literature for some time now [BR], [Hn], [Ha], [Ca], [CMM]. The volumes we introduce here (Section 2) are slightly different from the ones investigated so far, where it was always assumed (explicitly or implicitly) that the boundaries of volumes cross edges of underlying triangulations at midpoints. Instead, we require edges of our volumes to be perpendicular to, but not necessarily to be bisectors of, the corresponding edges in underlying triangulations. Using this type of volumes, we show in Section 3 that discrete solutions of Poisson equations satisfy the maximum principle. Furthermore, we prove convergence of discrete solutions for Dirichlet problems to the corresponding classical solutions in the H^1-norm (Theorem 3.5) and the L^2-norm (Theorem 3.6) under rather mild restrictions (i.e., regularity) on triangulations and volumes involved in the process of generating discrete solutions.

In Section 4 we show that if triangulations used to construct discrete solutions are close to being uniform, then discrete solutions will approximate the classical counterparts uniformly on compact subsets. This result is proved for both continuous and discontinuous boundary conditions, however, in the latter case we require some smoothness on domains involved in Dirichlet problems.

The question of how to generate "good" triangulations and volumes is addressed in Section 5. There we show that triangulations and volumes induced by circle packings have all the desired properties, i.e., regularity, provided some combinatorial (but not geometric) restrictions on tangency patters in circle packings, which, from practical point of view, are essentially always satisfied (Corollary 5.1).

We also comment on how a Dirichlet problem for a Poisson equation can be pulled back to a standard domain (e.g., the unit disk) using the discrete Riemann mapping theorem for circle packings (Theorem 5.2 and Corollary 5.4).

Finally, we prove that random walks induced by circle packings, which were introduced in [Du3] and [Du1], have a similar behavior to that of Brownian motion by showing that exit probabilities of a sufficiently dense circle packing filling a domain in the plain are close to corresponding exit probabilities of Brownian motion in that domain.

2. Triangulations and Volumes

We begin with a description of triangulations and associated volume-triangulations. Suppose T is a (finite) triangulation of a simply connected domain in the plane $\mathbf{R}^2$. We denote the set of vertices (nodes), edges, and triangles (faces) of T by T^0, T^1, and T^2, respectively. We also write IT^0 and ∂T^0 for the sets of interior and boundary vertices of T. We use ∂T for the (geometric) boundary of the set T. Also, the symbol $\sim$ is used to denote adjacent elements in T^0 or in T^2.

Now, T^* is said to be a *volume-triangulation* of T in $\mathbf{R}^2$ (i.e., a 2-cell dual triangulation) if the following holds: for every triangle $t \in T^2$ there is a unique point z_t inside it so that (1) z_t can be orthogonaly projected on each side of t, and (2) if t and t' are two adjacent triangles, then the segment $z_t z_{t'}$ joining z_t and $z_{t'}$ is perpendicular to and intersects the common side of t and t'. For $z \in IT^0$, V_z denotes the *volume* associated with z, i.e., a polygon bounded by edges $z_{t_1}z_{t_2}, z_{t_2}z_{t_3}, \ldots, z_{t_n}z_{t_1}$, where $t_1, \ldots, t_n$ are consecutive triangles of T with vertex z. If $z \in \partial T^0$, then V_z is a polygon bounded by edges zz'_{t_1}, $z'_{t_1}z_{t_1}, z_{t_1}z_{t_2}, z_{t_2}z_{t_3}, \ldots, z_{t_{m-1}}z_{t_m}, z_{t_m}z'_{t_m}, z'_{t_m}z$, where $t_1, \ldots, t_m$ are consecutive triangles of

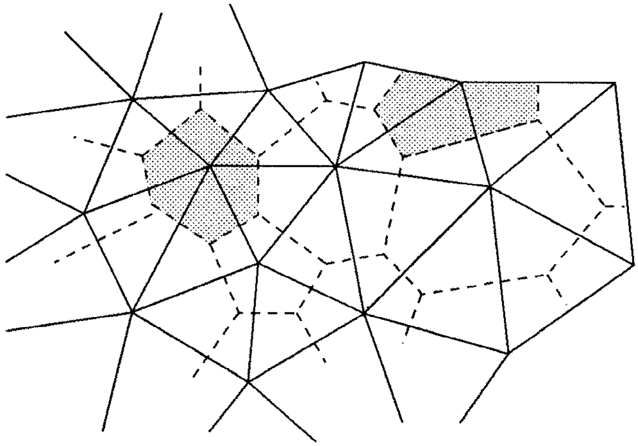

Fig. 1. A triangulation (solid lines) and an associated volume-triangulation (dashed lines). Two shaded polygons indicate an interior volume and a boundary volume.

T with vertex z (with t_1 and t_m being boundary triangles), and z'_{t_1} (respectively, z'_{t_m}) is the image point of the orthogonal projection of z_{t_1} (respectively, z_{t_m}) onto the boundary edge of t_1 (respectively, t_m) originated at z (see Fig. 1).

The regularity constant σ_T of a triangulation T (see [Ci]) is defined by

$$\sigma_T := \sup_{t\in T^2} \frac{\mathrm{diam}(t)}{\mathrm{in}(t)},$$

where $\mathrm{in}(t)$ is the radius of the inscribed circle of t and $\mathrm{diam}(t)$ is the diameter of t. A large value of σ_T indicates that T has some rather flat triangles. A family of triangulations $\{T_n\}$ is said to be *regular* if there exists σ such that $\sigma_{T_n} < \sigma$ for all n.

Similarly, we define the regularity constant of a volume-triangulation T^* by

$$\sigma_{T^*} := \sup_{t\in T^2} \frac{\mathrm{diam}(t)}{\mathrm{dist}(z_t, \partial t)},$$

where dist is the distance function. A family of volume-triangulations $\{T_n^*\}$ is said to be *regular* if $\{T_n\}$ is regular and $\sigma_{T_n^*} < \sigma^*$ for some $\sigma^* > 0$ and all n.

For every pair t and t' of neighboring triangles in T there is the volume $V_{t\cap t'}$ associated with their common edge $t\cap t'$: if z_i and z_j are the endpoints of $t\cap t'$, then $V_{t\cap t'}$ is built of two triangles $\triangle z_t z_{t'} z_i$ and $\triangle z_t z_{t'} z_i$ (see Fig. 2(a)). For future references, we remark that if $\rho_{t\cap t'} := \min\{\mathrm{in}(\triangle z_t z_{t'} z_i), \mathrm{in}(\triangle z_t z_{t'} z_j)\}$ and $\{T_n^*\}$ is regular, then there exists σ such that

$$\frac{\mathrm{dist}(z_t, z_{t'})}{\rho_{t\cap t'}} < \sigma, \qquad (\diamond)$$

for every $t, t' \in T_n^2$, $t \sim t'$, and all n.

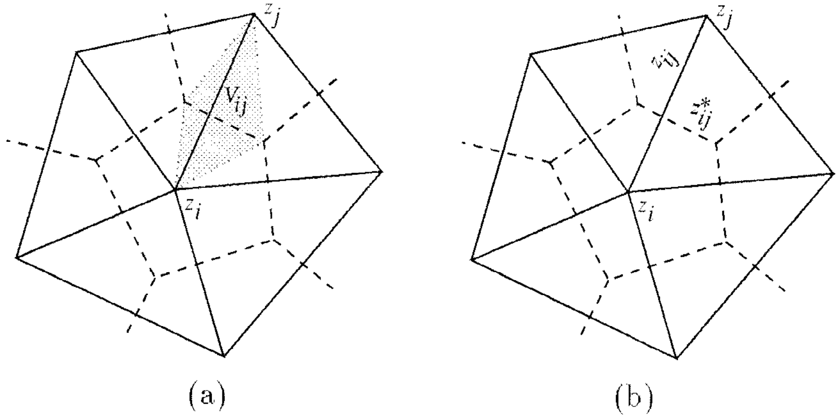

Fig. 2. (a) The volume V_{ij}, and (b) the segments z_{ij} and z^*_{ij}.

It is convenient in what follows to use the following notation: for $t \in T^2$, $|t|$ denotes the area of t; if V_z and $V_{t\cap t'}$ are volumes, then $|V_z|$ and $|V_{t\cap t'}|$ denote their areas, respectively. Furthermore, if $t, t' \in T^2$ are adjacent and their common side has the endpoints z_i and z_j, then we write $z_{ij} := t \cap t'$, $z^*_{ij} := z_t z_{t'}$, $\rho_{ij} := \rho_{t\cap t'}$, $V_{ij} := V_{t\cap t'}$, $|z_{ij}| = |z_i - z_j| := \text{dist}(z_i, z_j)$, and $|z^*_{ij}| := |z_t - z_{t'}|$ (Fig. 2(b)).

3. Finite Volume Method

In this section we describe a finite volume method. Let Ω be a domain in $\mathbf{R}^2$. By $H^k(\Omega)$, $0 \leq k$, we denote the standard kth Sobolev space, i.e., the set of functions in Ω with finite $\|\cdot\|_{H^k(\Omega)}$ norm, $\|u\|_{H^k(\Omega)} := (\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^2\, dx)^{1/2}$, where $D^\alpha u$ is a weak derivative of u, and α is a multi-index (for details, see [GT]). We write $|u|_{H^k(\Omega)} := (\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^2 dx)^{1/2}$ for the seminorm in $H^k(\Omega)$. Furthermore, we define $H^\infty(\Omega) := \{u : \|u\|_{L^\infty(\Omega)} < \infty\}$, where $\|u\|_{L^\infty(\Omega)}$ denotes the essential supremum of u in Ω.

We now introduce discrete versions of the above (semi)norms. Let T be a triangulation of a domain in $\mathbf{R}^2$. Denote the set of real-valued functions defined on vertices of T by $\Sigma^0(T)$, and the set of continuous functions w: $T \to \mathbf{R}$ that are linear on each $t \in T^2$ by $\Sigma^1(T)$. If $w \in \Sigma^0(T)$, then its linear extension is denoted by $\widehat{w} \in \Sigma^1(T)$.

We first introduce the following inner product, and discrete H^0- and sup-norms in $\Sigma^0(T)$: for $u, w \in \Sigma^0(T)$,

$$\begin{aligned}(u, w)_T &:= \sum_{z\in T^0} u(z)w(z)|V_z|,\\ \|u\|^2_{0,T} &:= (u, u)_T,\\ \|u\|_{\infty,T} &:= \sup_{z\in T^0} |u(z)|.\end{aligned}$$

The discrete H^1 seminorm and norm in $\Sigma^0(T)$ are defined by

$$|u|_{1,T}^2 := \sum_{z_{ij}\in T^1} (D_{ij}u)^2 |V_{ij}|, \qquad \text{where} \quad D_{ij}u := \frac{u(z_j) - u(z_i)}{|z_{ij}|},$$
$$\|u\|_{1,T}^2 := \|u\|_{0,T}^2 + |u|_{1,T}^2.$$

For future reference we make the following observation:

Remark 3.1. $|u|_{1,T}^2 = \frac{1}{2}\sum_{z_{ij}\in T^1}(|z_{ij}^*|/|z_{ij}|)|u(z_i) - u(z_j)|^2.$

Finally, we extend the definitions of the above discrete (semi)norms to functions defined in T as follows: if $w\colon T \to \mathbf{R}$, then $\|u\|_{\infty,T} := \|u_{|T^0}\|_{\infty,T}$ and $|u|_{1,T} := |u_{|T^0}|_{1,T}$, where $u_{|T^0}$ is the restriction of u to the set T^0. If ω is a subset of T, then $\|u\|_{\infty,\omega} := \sup_{z\in\omega\cap T^0}|u(z)|$.

For functions in $\Sigma^1(T)$, the classical and discrete definitions of H^1-seminorms are closely related in the following way (for a proof see, e.g., [BR]):

Proposition 3.2. *There exists a constant $C = C(\sigma_T)$ depending only on the regularity constant σ_T of T such that, for $u \in \Sigma^1(T)$,*

$$\frac{1}{C}|u|_{H^1(T)} \le |u|_{1,T} \le C|u|_{H^1(T)}.$$

We now introduce an operator whose domain is $H^2(T)\cup\Sigma^1(T)\cup\Sigma^0(T)$ and the range is the space of real-valued functions defined over IT^0. If $w \in H^2(T)\cup\Sigma^1(T)$, then

$$A_T w(z) := -\frac{1}{|V_z|}\int_{\partial V_z} \nabla w\cdot\overrightarrow{\eta}\, ds \qquad \text{for} \quad z \in IT^0,$$

where $\overrightarrow{\eta}$ denotes the outward unit normal vector on the boundary ∂V_z of V_z. If $w \in \Sigma^0(T)$, then $A_T w := A_T\widehat{w}$. We extend the operator A_T to $\bar{A}_T\colon H^2(T)\cup\Sigma^1(T)\cup\Sigma^0(T)\to\Sigma^0(T)$ by $\bar{A}_T w(z) := A_T w(z)$ if $z \in IT^0$ and $\bar{A}_T w(z) := 0$ for $z\in\partial T^0$.

Suppose that $f \in L^2(T)$, $\varphi \in C(\partial T)$, and u is the solution to the Dirichlet problem $-\triangle u = f$ in T and $u = \varphi$ on ∂T. Then the corresponding discrete problem is defined as follows:

$$\text{find} \quad w\colon T^0 \to \mathbf{R} \quad \text{such that} \quad \begin{cases} A_T w(z) = f_T(z) & \text{for} \quad z \in IT^0, \\ w(z) = \varphi(z) & \text{for} \quad \in \partial T^0, \end{cases}$$

where $f_T(z) := (1/|V_z|)\int_{V_z} f\,dx$.

Remark 3.3. 1. Notice that the discrete problem defined above is modeled on a classical approach where a solution of the equation $-\triangle u = f$ is found by replacing the differential equation by the integral condition: $-(1/|V|)\int_{\partial V}\nabla u\cdot\overrightarrow{\eta}\,ds = (1/|V|)\int_V f\,dx$ for every subset $V \subset \overline{\Omega}$ with Lipschitz boundary.

2. The discrete problem is a linear problem. In general, if $(F, \Phi) \in \mathbf{R}^{|IT^0|+|\partial T^0|}$, then the discrete Dirichlet problem, $A_T w = F$ in IT^0 and $w = \Phi$ on ∂T^0, has the following explicit formulation:

$$\begin{cases} \frac{1}{|V_{z_i}|} \sum_{z_j \sim z_i} \frac{|z_{ij}^*|}{|z_{ij}|}(w(z_i) - w(z_j)) = F(z_i) & \text{for} \quad z_i \in IT^0, \\ w(z) = \Phi(z) & \text{for} \quad z \in \partial T^0. \end{cases} \qquad (\star)$$

Solutions of the above linear problem have the following important property.

Maximum Principle. *If $F \geq 0$, then a solution w of $(\star)$ attains its minimum on ∂T^0. In particular, if $\Phi \geq 0$, then $w \geq 0$.*

Proof. From the equations in $(\star)$ involving interior vertices and the assumption that $F \geq 0$, it follows that if w attains its global minimum at an interior vertex then w must be constant, in particular, attaining minimum on ∂T^0. □

Remark 3.4. 1. From the Maximum Principle one obtains that the linear system of equations $(\star)$ is always uniquely solvable.

2. The above Maximum Principle can also be derived from a probabilistic interpretation of equations $(\star)$ as discussed in the last section of this paper.

The next result gives some estimates on an error between the classical solution of a Dirichlet problem and its discrete counterpart; the result is essentially due to Cai and coworkers [Ca], [CMM]. Differences are in the assumptions on families of triangulations and boundary conditions; we do not require sides of volumes in T_n^* to be bisectors of sides of triangles in T_n nor do we impose any conditions on angles of triangles of T_n, and the boundary of T_n does not need to coincide with the boundary of the domain considered.

Theorem 3.5. *Suppose $u \in H^2(\Omega)$ is a solution of $-\triangle u = f$ in Ω, $f \in L^2(\Omega)$. Let $\{T_n\}$ be a regular family of triangulations with $T_n \subseteq \Omega$ and $\mu_n \to 0$ as $n \to \infty$, where $\mu_n = \sup_{t \in T_n} \operatorname{diam}(t)$. Assume that $\{T_n^*\}$ is the corresponding family of volume-triangulations of $\{T_n\}$. Denote by u_n the discrete solution, in T_n, of*

$$\begin{cases} A_{T_n} w(z) = f_{T_n}(z), & z \in IT_n^0, \\ w(z) = u(z), & z \in \partial T_n^0. \end{cases}$$

Then

$$|u - u_n|_{1,T_n} \leq C \mu_n |u|_{H^2(\Omega)},$$

where C is a constant that depends only on the regularity of families $\{T_n\}$ and $\{T_n^*\}$.

Proof. Since the proof requires only minor modifications to the one in [CMM], we outline here the major steps/differences, and for details the reader is refered to [CMM].

First, we notice that, for $w \in \Sigma^0(T_n)$, one has $(\bar{A}w, w)_{T_n} = |w|^2_{1,T_n}$, that follows from direct calculations:

$$\begin{aligned}
(\bar{A}w, w)_{T_n} &= \sum_{z_i \in IT_n^0} w(z_i) \left(\sum_{z_j \sim z_i} - \int_{z_{ij}^*} \nabla \widehat{w} \cdot \overrightarrow{\eta}\, ds \right) \\
&= \sum_{z_i \in IT_n^0} w(z_i) \left(\sum_{z_j \sim z_i} \frac{|z_{ij}^*|}{|z_{ij}|} (w(z_i) - w(z_j)) \right) \\
&= \sum_{z_i \in T_n^0} w(z_i) \left(\sum_{z_j \sim z_i} \frac{|z_{ij}^*|}{|z_{ij}|} (w(z_i) - w(z_j)) \right) \\
&= \tfrac{1}{2} \sum_{z_{ij} \in T_n^1} \frac{|z_{ij}^*|}{|z_{ij}|} (w(z_i) - w(z_j))^2 = |w|^2_{1,T_n}.
\end{aligned}$$

Second, because $u \in H^2(\Omega)$, the Sobolev embedding theorem implies that $u \in C(\Omega)$ (and $u \in C(\bar{\Omega})$ if Ω has the exterior cone property (see [Ad] and [GT])). Next, define $e_n := u - u_n$ and $e_I^n := u - u_I^n$, where $u_I^n := \widehat{u_{|T_n^0}}$, i.e., u_I^n is the linear interpolant of u_n over T_n. Then, from the definition of discrete solutions and the fact that $u_n = u_I^n$ on ∂T_n^0, we obtain

$$\begin{aligned}
|e_n|^2_{1,T_n} = (\bar{A}e_n, e_n)_{T_n} &= \sum_{z_{ij} \in T_n^1} (e_n(z_j) - e_n(z_i)) \left(- \int_{z_{ij}^*} \nabla e_I^n \cdot \overrightarrow{\eta}\, ds \right) \\
&\leq |e_n|_{1,T_n} \left(\sum_{z_{ij} \in T_n^1} \frac{|z_{ij}^*|}{|z_{ij}|} \left(- \int_{z_{ij}^*} \nabla e_I^n \cdot \overrightarrow{\eta}\, ds \right)^2 \right)^{1/2},
\end{aligned}$$

that implies

$$|e_n|_{1,T_n} \leq \left(\sum_{z_{ij} \in T_n^1} \frac{|z_{ij}^*|}{|z_{ij}|} \left(- \int_{z_{ij}^*} \nabla e_I^n \cdot \overrightarrow{\eta}\, ds \right)^2 \right)^{1/2}.$$

Now, from the regularity of families $\{T_n\}$ and $\{T_n^*\}$, and the property $(\diamond)$, we obtain (exactly the same way as in Lemma 3 of [CMM])

$$\left| \int_{z_{ij}^*} \nabla e_I^n \cdot \overrightarrow{\eta}\, ds \right| \leq C |z_{ij}|^{5/2} |z_{ij}^*|^{1/2} \rho_{ij}^{-2} |u|_{H^2(V_{ij})},$$

where C is a constant depending only on the regularity of families $\{T_n\}$ and $\{T_n^*\}$. Hence

$$|e_n|_{1,T_n} \leq \left(\sum_{z_{ij} \in T_n^1} C^2 \sigma^4 |z_{ij}|^2 |u|^2_{H^2(V_{ij})} \right)^{1/2} \leq C\sigma^2 \mu_n |u|_{H^2(\Omega)},$$

where σ is a constant as in $(\diamond)$. □

We now investigate the L^2-convergence of discrete solutions. Suppose Ω is a Jordan domain and φ is a continuous function on $\partial\Omega$. Suppose, further, that there is a neighborhood $\Omega^\varepsilon \subset \Omega$ of $\partial\Omega$ and a method for construction of a continuous function $\bar{\varphi}\colon\ \Omega^\varepsilon \cup \partial\Omega \mapsto \mathbf{R}$ such that $\bar{\varphi} = \varphi$ on $\partial\Omega$. (From Tietze's theorem, we know that such an extension always exists, however, it may be hard to construct it in a manageable way.) For example, when $\partial\Omega$ is C^2 (i.e., a twice continuously differentiable curve) then there exists ε such that when $\mathrm{dist}(z, \partial\Omega) < \varepsilon$ then there is a unique point $z_\partial \in \partial\Omega$ with $\mathrm{dist}(z, z_\partial) = \mathrm{dist}(z, \partial\Omega)$, and $\bar{\varphi}$ can be defined by a projection, i.e., $\bar{\varphi}(z) := \varphi(z_\partial)$.

Let $f \in L^2(\Omega)$ and let u be the solution of the Dirichlet problem

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{$*$}$$

If T is a triangulation with the boundary as a Jordan curve, and $T \subseteq \bar{\Omega}$, $\partial T \subset \Omega^\varepsilon \cup \partial\Omega$, and there is an associated volume-triangulation T^*, then we define the corresponding approximate solution u_T of the above continuous problem by

$$\begin{cases} \bar{A}_T u_T(z) = f_T(z) & \text{for} \quad z \in IT^0, \\ u_T(z) = \bar{\varphi}(z) & \text{for} \quad z \in \partial T^0, \end{cases} \tag{$**$}$$

i.e., the system of equations $(**)$ is just a generalization of the earlier definition to a case where the boundary of T does not coincide with that of Ω.

The following result addresses a question of the L^2-convergence of discrete solutions to the classical one.

Theorem 3.6. *Let Ω be a Jordan domain with C^2-boundary, $\varphi \in C(\partial\Omega)$, and $f \in L^2(\Omega)$. Suppose $\bar{\varphi}$ is a continuous extension of φ to an inside neighborhood of $\partial\Omega$. Denote by u the solution to the Dirichlet problem $(*)$. Assume that $\{T_n\}$ is a family of triangulations such that, for each n, T_n is simply connected, $T_n \subseteq \bar{\Omega}$, $T_n \to \Omega$ as $n \to \infty$ (i.e., sets T_n exhaust Ω), and $\mu_n \to 0$, where μ_n is the mesh size of T_n. Suppose, further, that $\{T_n^*\}$ is an associated family of volume-triangulations, which is regular. For each n, let u_n be the discrete solution of $(**)$ in T_n. Then* $\lim_{n\to\infty} \|u - u_n\|_{L^2(T_n)} = 0$.

Before we give a proof, we make a few remarks.

Remark 3.7. 1. The above result is true for any Jordan domain, not necessarily with a C^2-boundary. By adopting techniques used in the next section together with the proof below one can give a proof of the general case. However, it should be noted that as $\partial\Omega$ gets more bizarre, it is much harder to get a good construction for an extension map $\bar{\varphi}$.

2. The above result can also be extended to Dirichlet problems for nonsimply connected domains.

3. The sequence of maps u_n is bounded by the Maximum Principle. We conjecture that it is in fact a locally equicontinuous family of mappings. If this is true then $\{u_n\}$ forms a normal family, and hence the u_n's converge uniformly on compacta of Ω to some continuous function $\tilde{u}$. Then it would follow from the above theorem that $\tilde{u}$ must be equal to u, i.e., $u_n \to u$ uniformly on compact subsets of Ω.

Proof of Theorem 3.6. Let $\varepsilon > 0$ and let $\psi \in C^\infty(\partial\Omega)$ such that $|\psi - \varphi|_{\infty,\Omega} < \varepsilon$. Write $\tilde{u}$ for the solution of $(*)$ with boundary condition ψ. Denote by $\tilde{u}_I^n$ the linear interpolant of $\tilde{u}$ in T_n, i.e., $\tilde{u}_I^n := \widehat{\tilde{u}_{|T_n^0}}$. Also, denote by $\tilde{u}_n$ the solution of $(**)$ in T_n with boundary condition $u_{|T_n^0}$. Then

$$\|u-u_n\|_{L^2(T_n)} \le \|u-\tilde{u}\|_{L^2(T_n)} + \|\tilde{u}-\tilde{u}_I^n\|_{L^2(T_n)} + \|\tilde{u}_I^n-\tilde{u}_n\|_{L^2(T_n)} + \|\tilde{u}_n-u_n\|_{L^2(T_n)}. \quad (1)$$

We are going to give estimates on terms on the right-hand side of the above inequality. The smoothness assumptions on $\partial\Omega$ and ψ imply [GT, Theorem. 8.12] that $\tilde{u} \in H^2(\Omega)$. Recall that if a function $w \in H^1(\Omega)$ is such that $w_{|\partial\Omega} = 0$, then the following Poincaré inequality [GT] holds:

$$\|w\|_{L^2(\Omega)} \le \left(\frac{|\Omega|}{\pi}\right)^{1/2} |w|_{H^1(\Omega)}. \quad (2)$$

Because $\tilde{u}_I^n - \tilde{u}_n = 0$ on ∂T_n, from the Poincaré ineqality, Theorem 3.5, and Proposition 3.2, it follows that

$$\begin{aligned}\|\tilde{u}_I^n - \tilde{u}_n\|_{L^2(T_n)} &\le \left(\frac{|T_n|}{\pi}\right)^{1/2} |\tilde{u}_I^n - \tilde{u}_n|_{H^1(T_n)} \le C\left(\frac{|\Omega|}{\pi}\right)^{1/2} |\tilde{u}_I^n - \tilde{u}_n|_{H^1(T_n)} \\ &\le C\left(\frac{|\Omega|}{\pi}\right)^{1/2} \mu_n |\tilde{u}|_{H^2(\Omega)}.\end{aligned}$$

Since $\partial\Omega$ is C^2, we get [GT, Theorem 9.30] that $u, \tilde{u} \in C(\bar{\Omega})$. As $u - \tilde{u}$ is a harmonic function, we obtain that $|u-\tilde{u}|_{\infty,\Omega} \le |u-\tilde{u}|_{\infty,\partial\Omega} \le \varepsilon$. Thus

$$\|u-\tilde{u}\|_{L^2(T_n)} \le \|u-\tilde{u}\|_{L^2(\Omega)} \le |\Omega|\varepsilon. \quad (3)$$

The definitions of $\tilde{u}_n$ and u_n together with the fact that $|u-\tilde{u}|_{\infty,\Omega} \le \varepsilon$ imply that $|\tilde{u}_n - u_n|_{\infty,\partial T_n} \le 2\varepsilon$ for all large n. By applying the Maximum Principle to discrete solutions $\tilde{u}_n$ and u_n, we obtain that $|\tilde{u}_n - u_n|_{\infty,T_n} \le \varepsilon$ for all large n. Hence

$$\|\tilde{u}_n - u_n\|_{L^2(T_n)} \le 2\varepsilon|\Omega|, \quad (4)$$

for large n. Finally, from $\tilde{u} \in H^2(\Omega)$ it follows [Ca, Theorems 3.1.6 and 3.2.1] that

$$\lim_{n\to\infty} \|\tilde{u} - \tilde{u}_I^n\|_{L^2(T_n)} = 0. \quad (5)$$

Thus, by combining (1)–(5) we obtain the assertion of the theorem. □

4. L^∞-Convergence

In this section we show that, under some additional conditions on families of triangulations involved in the construction of discrete solutions, we obtain convergence in sup-norm on compact subsets.

Let $\{T_n\}$ and $\{T_n^*\}$ be regular families of triangulations and volume-triangulations. Throughout this section we assume in addition that $\{T_n\}$ is *quasi-uniform*, i.e., there exists a constant σ such that

$$\frac{\sup_{t\in T_n}\operatorname{diam}(t)}{\inf_{t\in T_n}\operatorname{diam}(t)} \le \sigma,$$

for every n (see [Ci]). As in the previous section, suppose Ω is a Jordan domain, $f \in L^2(\Omega)$, and $\varphi \in C(\Omega)$. We assume that we also have a continuous map $\bar{\varphi}$ defined inside Ω in some neighborhood of $\partial\Omega$, which is an extension of φ. Let u be the solution of $(*)$. We denote by u_n the discrete solution of $(**)$ for $T := T_n$. Then the main result is the following theorem.

Theorem 4.1. *If $\{T_n\}$ and $\{T_n^*\}$ are regular families of triangulations and associated volume-triangulations, $\{T_n\}$ is quasi-uniform, the sets T_n exhaust Ω from inside, and $\mu_n = \sup_{t\in T_n}\operatorname{diam}(t) \to 0$, then the sequence of maps u_n converges uniformly on compact subsets of Ω to u.*

Remark 4.2. 1. As we have pointed out in Remark 3.7, we believe that the conclusion of the theorem is true without quasi-uniform condition on $\{T_n\}$. We hope to resolve this issue in a sequel.

2. There are related results that address the convergence in sup-norm in Chapter 3.3 of [Ci] and in [Hn]. The main differences are that the boundaries of T_n's are not that rigorously associated with $\partial\Omega$ here as they are in [Ci] and [Hn], and the volumes here are different from the ones in [Hn]. This allows for consideration of a broader class of domains but yields loss in estimates for the rate of convergence.

The proof of Theorem 4.1 is given in a sequence of lemmas, where the assertion of the theorem is first proved for C^2-domains and then the general case is split into two parts: the case of harmonic solutions and the case with zero boundary condition.

Lemma 4.3. *Suppose Ω is a C^2-domain, $f \in L^2(\Omega)$, and $\varphi \in C(\Omega)$. Then under the assumptions of Theorem* 4.1, $\|u - u_n\|_{\infty,T_n} \to 0$.

Proof. Let $\varepsilon > 0$, and let $\varphi^\varepsilon \in C(\mathbf{R}^2)$ be such that $\|\varphi - \varphi^\varepsilon\|_{L^\infty(\partial\Omega)} < \varepsilon$. Define u^ε to be the solution of $(*)$ with boundary condition φ^ε. Then, since $\varphi^\varepsilon \in C(\mathbf{R}^2)$, it follows [GT, Theorem 8.12] that $u^\varepsilon \in H^2(\Omega) \cap C(\bar{\Omega})$. Let u_n^ε be the discrete solution of $(**)$ in T_n for boundary condition $u_n^\varepsilon(z) = u^\varepsilon(z)$, $z \in \partial T_n^0$. By applying the Maximum Principle to the discrete solutions u_n and u_n^ε, and to the classical solutions u and u^ε, we obtain the following inequalities:

$$\begin{aligned}
\|u - u_n\|_{\infty,T_n} &\le \|u - u^\varepsilon\|_{\infty,T_n} + \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|u_n^\varepsilon - u_n\|_{\infty,T_n} \\
&\le \|u - u^\varepsilon\|_{L^\infty(\Omega)} + \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|u_n^\varepsilon - u_n\|_{\infty,\partial T_n} \\
&= \|\varphi - \varphi^\varepsilon\|_{L^\infty(\partial\Omega)} + \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|u_n^\varepsilon - u_n\|_{\infty,\partial T_n} \\
&\le \varepsilon + \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|\varphi^\varepsilon - \bar{\varphi}\|_{\infty,\partial T_n}.
\end{aligned}$$

Since $\bar{\varphi}$ is a continuous extension of φ, and $\partial T_n \to \partial\Omega$ as $n \to \infty$, we get that $\lim_{n\to\infty} \|\bar{\varphi} - \varphi^\varepsilon\|_{\infty,\partial T_n} < 2\varepsilon$. To give an estimate on the term $\|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n}$ we need the following result which is due to Oganesyan nad Rukhovets [OR, pp. 74–77]: there exists a constant $C = C(\sigma)$, depending only on σ, such that, for every triangulation T that is regular and quasi-uniform with corresponding constants no bigger that σ and every $w \in \Sigma^1(T)$ with $w_{|\partial T} = 0$, we have $\|w\|_{L^\infty(T)} \le C|\log \mu_T|^{1/2}\|w\|_{H^1(T)}$, where $\mu_T := \sup_{t\in T} \operatorname{diam}(t)$. Now, from Theorem 3.5, the Poincaré inequality, Proposition 3.2, and the above result, we obtain

$$\begin{aligned}\|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} &= \|\widehat{u^\varepsilon_{|T_n^0}} - \widehat{u_n^\varepsilon}\|_{L^\infty(T_n)} \le C|\log \mu_n|^{1/2}\|\widehat{u^\varepsilon_{|T_n^0}} - \widehat{u_n^\varepsilon}\|_{H^1(T_n)}\\ &\le C'|\log \mu_n|^{1/2}|\widehat{u^\varepsilon_{|T_n^0}} - \widehat{u_n^\varepsilon}|_{H^1(T_n)} \le \tilde{C}|\log \mu_n|^{1/2}|u^\varepsilon - u_n^\varepsilon|_{1,T_n}\\ &\le \tilde{C}|\log \mu_n|^{1/2}\mu_n|u^\varepsilon|_{H^2(\Omega)},\end{aligned}$$

where $C, C', \tilde{C}$, and $\tilde{C}'$ are just constants independent of u^ε or the mesh size of T_n. Thus $\lim_{n\to\infty} \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} = 0$. Hence $\lim_{n\to\infty} \|u - u_n\|_{\infty,T_n} \le 3\varepsilon$, and as ε is arbitrary, this completes the proof. □

We now look into the harmonic part of the solution u.

Lemma 4.4. *Suppose Ω is a Jordan domain, $f \equiv 0$, and $\varphi \in C(\Omega)$. Then under the assumptions of Theorem* 4.1, $\|u - u_n\|_{\infty,T_n} \to 0$.

Proof. Denote by $\tau\colon \Omega \to \mathbf{D}$ a Riemann mapping, where $\mathbf{D}$ is the unit disk. Let $\{\Omega_\varepsilon\}$ be a sequence of C^2 Jordan domains such that $\bar{\Omega} \subset \Omega_\varepsilon$, $\bar{\Omega}_{\varepsilon'} \subset \Omega_\varepsilon$ for $\varepsilon' < \varepsilon$, $\bigcap \Omega_\varepsilon = \Omega$, and the boundary of Ω_ε converges to $\partial\Omega$ in the sense of Fréchet (see p. 27 of [LV] and [Wa]) as $\varepsilon \to 0$. We define $\tau_\varepsilon\colon \Omega_\varepsilon \to \mathbf{D}$ to be the Riemann mapping such that $\tau_\varepsilon(\tau^{-1}(0)) = 0$ and $\tau_\varepsilon(\tau^{-1}(\frac{1}{2})) > 0$. Then $\tau_\varepsilon \to \tau$ uniformly in $\bar{\Omega}$ (see [Wa] or [Du2]), and hence $\tau^{-1} \circ \tau_\varepsilon \to id$ uniformly in $\bar{\Omega}$.

Let $u^\varepsilon := u \circ \tau^{-1} \circ \tau_\varepsilon : \Omega_\varepsilon \to \mathbf{R}$. Then $\triangle u^\varepsilon = 0$ in Ω_ε, $u^\varepsilon \in H^2(\Omega)$, and $\|u - u^\varepsilon\|_{L^\infty(\bar{\Omega})} \to 0$ as $\varepsilon \to 0$.

Let u_n^ε be the discrete solution of $(**)$ in T_n with $f_{T_n} \equiv 0$ and boundary condition $u_n^\varepsilon(z) = u^\varepsilon(z)$ for $z \in \partial T_n^0$. Using the Maximum Principle we obtain the following estimates:

$$\begin{aligned}\|u - u_n\|_{\infty,T_n} &\le \|u - u^\varepsilon\|_{\infty,T_n} + \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|u_n^\varepsilon - u_n\|_{\infty,T_n}\\ &\le \|u - u^\varepsilon\|_{\infty,T_n} + \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|u_n^\varepsilon - u_n\|_{\infty,\partial T_n}\\ &\le \|u - u^\varepsilon\|_{L^\infty(\partial\Omega)} + \|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|u_n^\varepsilon - u^\varepsilon\|_{\infty,\partial T_n}\\ &\qquad + \|u^\varepsilon - u\|_{\infty,\partial T_n} + \|u - u_n\|_{\infty,\partial T_n}\\ &\le 2\|u - u^\varepsilon\|_{L^\infty(\partial\Omega)} + 2\|u^\varepsilon - u_n^\varepsilon\|_{\infty,T_n} + \|u - u_n\|_{\infty,\partial T_n}.\end{aligned}$$

Let $\delta > 0$. Recall that $\bar{\varphi}$ is a continuous extension of φ near $\partial\Omega$. Hence $\bar{\varphi} - u$ is a continuous function in some neighborhood of $\partial\Omega$, inside of Ω, and $\bar{\varphi} - u = 0$ on $\partial\Omega$. Hence, there is some neighborhood of $\partial\Omega$ such that, for any point z in this neighborhood,

$|\bar{\varphi}(z) - u(z)| < \delta$. Therefore, from the boundary condition on u_n's it follows that, for all sufficiently large n, $\|u - u_n\|_{\infty,\partial T_n} < 2\delta$. Next, we choose ε_δ to be small enough so that $\|u^{\varepsilon_\delta} - u\|_{L^\infty(\partial\Omega)} < \delta$; this is guaranted by the fact that $\|u - u^\varepsilon\|_{L^\infty(\bar{\Omega})} \to 0$. Then Lemma 4.2 and the fact that $\partial\Omega_{\varepsilon_\delta}$ is C^2 imply that $\lim_{n\to\infty} \|u^{\varepsilon_\delta} - u_n^{\varepsilon_\delta}\|_{\infty,T_n} = 0$. Hence we obtain that $\lim_{n\to\infty} \|u - u_n\|_{\infty,T_n} \le 5\delta$. □

In the next result we deal with the case of homogeneous boundary data.

Lemma 4.5. *Suppose* Ω *is a Jordan domain,* $f \in L^2(\Omega)$, *and* $\varphi \equiv 0$. *Then under the assumptions of Theorem* 4.1, $u_n \to u$ *uniformly on compacta of* Ω.

Proof. Let $\{\Omega_\varepsilon^+\}$ and $\{\Omega_\varepsilon^-\}$ be sequences of C^2 Jordan domains such that $\bar{\Omega} \subset \Omega_\varepsilon^+$, $\bar{\Omega}_{\varepsilon'}^+ \subset \Omega_\varepsilon^+$ for $\varepsilon' < \varepsilon$, $\bigcap \Omega_\varepsilon^+ = \Omega$, and $\bar{\Omega} \supset \Omega_\varepsilon^-$, $\bar{\Omega}_{\varepsilon'}^- \supset \Omega_\varepsilon^-$ for $\varepsilon' < \varepsilon$, $\bigcup \Omega_\varepsilon^- = \Omega$. Denote by $\bar{f}$ the extension of f which is 0 in $\mathbf{R}^2 \backslash \Omega$. We write $u^{\varepsilon+}$ for the solution $-\triangle u^{\varepsilon+} = \bar{f}$ in Ω_ε^+ and $u^{\varepsilon+} = 0$ on $\partial\Omega_\varepsilon^+$. Similarly, we denote by $u^{\varepsilon-}$ the solution $-\triangle u^{\varepsilon-} = f$ in Ω_ε^- and $u^{\varepsilon-} = 0$ on $\partial\Omega_\varepsilon^-$.

We also introduce the corresponding discrete solutions as follows. Let $u_n^{\varepsilon+}$ be the solution of $(**)$ in T_n with boundary condition $u_n^{\varepsilon+}(z) = u^{\varepsilon+}(z)$ for $z \in \partial T_n^0$. Denote by $T_{n,\varepsilon}$ the "intersection" of T_n with Ω_ε^-, i.e., the largest part of T_n contained in Ω_ε^- which is still a triangulation of a simply connected domain. Then let $u_n^{\varepsilon-}$ be the solution of $(**)$ in $T_{n,\varepsilon}$ with boundary condition $u_n^{\varepsilon-}(z) = 0$ for $z \in \partial T_{n,\varepsilon}^0$.

Suppose first that $0 \le f$. By applying the Maximum Principle to classical solutions we get

$$u \le u^{\varepsilon+} \quad \text{in } \Omega, \qquad \text{and} \qquad u^{\varepsilon-} \le u \quad \text{in } \Omega_\varepsilon^-.$$

Similarly, in the discrete setting we have

$$u_n(z) \le u_n^{\varepsilon+}(z) \quad \text{for } z \in T_n^0, \qquad \text{and} \qquad u_n^{\varepsilon-}(z) \le u_n(z) \quad \text{for } z \in T_{n,\varepsilon}^0; \qquad (\natural)$$

to obtain the first inequality above, we have used that $u \le u^{\varepsilon+}$ in Ω and $u_n^{\varepsilon+} = u^{\varepsilon+}$ on ∂T_n^0.

Since $\{u^{\varepsilon-}\}$ is an increasing sequence of functions as $\varepsilon \searrow 0$, and $u^{\varepsilon-} - u$ are harmonic, from Harnack's theorem we get that $u^{\varepsilon-} \to u$ uniformly on compact subsets of Ω. Similarly, as $\{u^{\varepsilon+}\}$ is a decreasing sequence of functions as $\varepsilon \searrow 0$, and $u - u^{\varepsilon+}$ are harmonic, we obtain that $u^{\varepsilon+} \to u$ uniformly on compacta of Ω.

From Lemma 4.3, we have that, for a fixed ε, $\|u^{\varepsilon-} - u_n^{\varepsilon-}\|_{\infty,\Omega_\varepsilon^-} \to 0$ as $n \to \infty$. Also from Lemma 4.3, the fact that, for a fixed ε, $u^{\varepsilon+} \in H^2(\Omega)$, and that $u_n^{\varepsilon+} = u^{\varepsilon+}$ on ∂T_n^0, we obtain that $\lim_{n\to\infty} \|u^{\varepsilon+} - u_n^{\varepsilon+}\|_{\infty,T_n} = 0$.

Now, let ω be a compact subset of Ω and let $\delta > 0$. If n is large enough so that $\omega \subset T_n$, then from $(\natural)$ we have

$$u_n(z) \ge u_n^{\varepsilon-}(z) = (u_n^{\varepsilon-}(z) - u^{\varepsilon-}(z)) + (u^{\varepsilon-}(z) - u(z)) + u(z)$$

and

$$u_n(z) \le u_n^{\varepsilon+}(z) = (u_n^{\varepsilon+}(z) - u^{\varepsilon+}(z)) + (u^{\varepsilon+}(z) - u(z)) + u(z),$$

for every $z \in \omega \cap T_n^0$. By choosing first ε_δ so that $\|u^{\varepsilon_\delta -} - u\|_{L^\infty(\omega)} < \delta$ and $\|u^{\varepsilon_\delta -} - u\|_{L^\infty(\omega)} < \delta$, and then $N = N(\delta, \varepsilon_\delta)$ large enough so that, for all $n \geq N$, $\|u^{\varepsilon_\delta -}(z) - u_n^{\varepsilon_\delta -}(z)\|_{\infty,\omega} < \delta$ and $\|u^{\varepsilon_\delta +}(z) - u_n^{\varepsilon_\delta +}(z)\|_{\infty,\omega} < \delta$, we get

$$\|u - u_n\|_{\infty,\omega} < 2\delta \qquad \text{for all} \quad n \geq N.$$

Since ω and δ are arbitrary, this shows that $u_n \to u$ uniformly on compact subsets of Ω in the case when $f \geq 0$. By symmetry, the same is true for $f \leq 0$, and the general case follows. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let u^h be the solution of $(*)$ for $f \equiv 0$, and let u^o be the solution of $(*)$ for $\varphi \equiv 0$. From the uniqueness of solutions it follows that $u = u^h + u^o$.

The same is true for discrete solutions. If u_n^h denotes the solution of $(**)$ for $f \equiv 0$, and u_n^o denotes the solution of $(**)$ for $\varphi \equiv 0$, then $u_n = u_n^h + u_n^o$. Now, the convergence $u_n \to u$ is an immediate consequence of Lemmas 4.3 and 4.4. □

Theorem 4.1 can, of course, be extended to include discontinuous boundary conditions. However, as more general cases are considered, it is getting much harder to define in a "practical" way boundary conditions for discrete solutions. We finish this section with a result related to discontinuous boundary conditions, which is applied in the next section.

Example 4.6. Suppose Ω is a C^2 Jordan domain, $f \in L^2(\Omega)$, and $\varphi = \chi_\gamma$, where γ is an arc in $\partial\Omega$ and $\chi_\gamma \colon \partial\Omega \to \{0, 1\}$ is the characteristic function of γ (i.e., $\chi_\gamma(z)$ is equal to 1 if $z \in \gamma$ and 0 otherwise). Denote by u the solution of $(*)$ with the above data. Let $\{T_n\}$ and $\{T_n^*\}$ be as in Theorem 4.1. Write u_n for the discrete solution of $(**)$ in T_n with the boundary condition $u_n(z) = \chi_\gamma(z_\partial)$, $z \in \partial T_n^0$, where $z \mapsto z_\partial$ is the projection of ∂T_n^0 to $\partial\Omega$ defined earlier for C^2-domains. Then $u_n \to u$ uniformly on compact subsets of Ω.

Proof. The proof is similar to that of Lemma 4.4. Let $\varphi^{\varepsilon+}, \varphi^{\varepsilon-} \in C(\partial\Omega)$ be such that $\varphi^{\varepsilon-} \leq \chi_\gamma \leq \varphi^{\varepsilon+}$, and the linear measure of sets $\{z \in \partial\Omega : |\varphi^{\varepsilon+}(z) - \chi_\gamma(z)| + |\varphi^{\varepsilon-}(z) - \chi_\gamma(z)| > 0\}$ goes to 0 as $\varepsilon \to 0$. In other words, $\varphi^{\varepsilon+}$ and $\varphi^{\varepsilon-}$ are two continuous "step" functions on $\partial\Omega$ that approximate χ_γ from above and below, respectively.

We define $u^{\varepsilon+}$ to be the solution of $-\triangle u^{\varepsilon+} = f$ in Ω and $u^{\varepsilon+} = \varphi^{\varepsilon+}$ on $\partial\Omega$. Similarly, we write $u^{\varepsilon-}$ for the solution of $-\triangle u^{\varepsilon-} = f$ in Ω and $u^{\varepsilon-} = \varphi^{\varepsilon-}$ on $\partial\Omega$. Then, from Harnak's theorem, it follows that $u^{\varepsilon+} \to u$ and $u^{\varepsilon-} \to u$ uniformly on compact subsets of Ω as $\varepsilon \to 0$.

Now let $u_n^{\varepsilon+}$ and $u_n^{\varepsilon-}$ be corresponding discrete solutions of $(**)$ in T_n with boundary conditions $u_n^{\varepsilon+} = \varphi^{\varepsilon+}$ on ∂T_n^0 and $u_n^{\varepsilon-} = \varphi^{\varepsilon-}$ on ∂T_n^0, respectively. Then, from the Maximum Principle for discrete solutions, we obtain

$$u_n(z) \geq u_n^{\varepsilon-}(z) = (u_n^{\varepsilon-}(z) - u^{\varepsilon-}(z)) + (u^{\varepsilon-}(z) - u(z)) + u(z)$$

and

$$u_n(z) \le u_n^{\varepsilon+}(z) = (u_n^{\varepsilon+}(z) - u^{\varepsilon+}(z)) + (u^{\varepsilon+}(z) - u(z)) + u(z),$$

for $z \in T_n^0$. As in the proof of Lemma 4.5, it now follows from the above inequalities and convergence of their terms in brackets to 0 on compact subsets, that, for every compact subset ω of Ω, $\|u - u_n\|_{\infty,\omega} \to 0$ as $n \to \infty$. □

5. Circle Packings and Random Walks

As we mentioned in the Introduction, this paper was motivated by the results in [Du1] and [Du3], where discrete harmonic functions for circle packings were introduced. In this section we discuss connections among circle packings, volume-triangulations, and random walks. We show how to generate triangulations and associate with them volumes for domain approximation by means of circle packings. We also describe how a Dirichlet problem from a reasonable domain can be pulled back to a standard domain, such as the unit disk, using the discrete Riemann mapping theorem.

We begin with a definition of circle packings (see also [BeS1], [BoS], [Du1], and [RS]). Let $\mathbb{K}$ be a simplicial 2-complex that is simplicially isomorphic to a triangulation of a closed disk in $\mathbf{R}^2$. We assume that $\mathbb{K}$ carries an orientation (induced, for example, from $\mathbf{R}^2$). Denote by $\mathbb{K}^0$, $I\mathbb{K}^0$, $\partial\mathbb{K}^0$, $\mathbb{K}^1$, and $\mathbb{K}^2$ the sets of vertices, interior vertices, boundary vertices, edges, and faces of $\mathbb{K}$, respectively. A collection $\mathcal{P} = \{\mathcal{C}_{\mathcal{P}}(\zeta)\}_{\zeta\in\mathbb{K}^0}$ of circles in $\mathbf{R}^2$ is said to be a *circle packing* for $\mathbb{K}$ if for every face $\langle\zeta_1, \zeta_2, \zeta_3\rangle$ in $\mathbb{K}$ with the vertices ζ_1, ζ_2, and ζ_3, listed in positive order, $\langle\mathcal{C}_{\mathcal{P}}(\zeta_1), \mathcal{C}_{\mathcal{P}}(\zeta_2), \mathcal{C}_{\mathcal{P}}(\zeta_3)\rangle$ is a triple of mutually and externally tangent circles in $\mathbf{R}^2$ listed in positive order (in $\mathbf{R}^2$) (see Fig. 3). We remark here that for any $\mathbb{K}$ there is a continuum family of associated circle packings, and any of such packings is uniquely determined by values of radii of boundary circles up to isometries (see [Du1], [BeS2], and [CdV]).

Assumption. Unless stated otherwise, we assume from now on that if $\mathcal{P}$ is a circle packing, then all circles in $\mathcal{P}$ have disjoint interiors.

If $\mathcal{P}$ is a circle packing for $\mathbb{K}$, then the *carrier* carr($\mathcal{P}$) of $\mathcal{P}$ is the collection $\{\langle\int_{\mathcal{P}}(\zeta_1), \int_{\mathcal{P}}(\zeta_2), \int_{\mathcal{P}}(\zeta_3)\rangle : \langle\zeta_1, \zeta_2, \zeta_3\rangle \in \mathbb{K}^2\}$ of triangles in $\mathbf{R}^2$, where $\int_{\mathcal{P}}(\zeta)$ denotes the center of the circle in $\mathcal{P}$ associated with vertex $\zeta \in \mathbb{K}^0$. It follows from our assumption about disjointness of interiors of circles that carr($\mathcal{P}$) is in fact a (piecewise linear) triangulation of a simply connected domain in $\mathbf{R}^2$, and it is simplicially isomorphic to the complex $\mathbb{K}$.

We now describe the volume-triangulation carr$(\mathcal{P})^*$ that corresponds to the triangulation carr($\mathcal{P}$). To do this, it is sufficient to define a point z_t for every triangle t in carr($\mathcal{P}$). If $t = \langle\int_{\mathcal{P}}(\zeta_1), \int_{\mathcal{P}}(\zeta_2), \int_{\mathcal{P}}(\zeta_3)\rangle$, then we define z_t to be the radical center of circles $\mathcal{C}_{\mathcal{P}}(\zeta_1)$, $\mathcal{C}_{\mathcal{P}}(\zeta_2)$, and $\mathcal{C}_{\mathcal{P}}(\zeta_3)$. (For more information, the reader is refered to [Du3], [Co], and [Ya].) Equivalently, z_t can be described as the center of the inscribed circle of t (see Fig. 4(a)). Then the volume V_z, $z = \int_{\mathcal{P}}(\zeta)$, is a polygon circumscribed on $\mathcal{C}_{\mathcal{P}}(\zeta)$, as in Fig. 4(b).

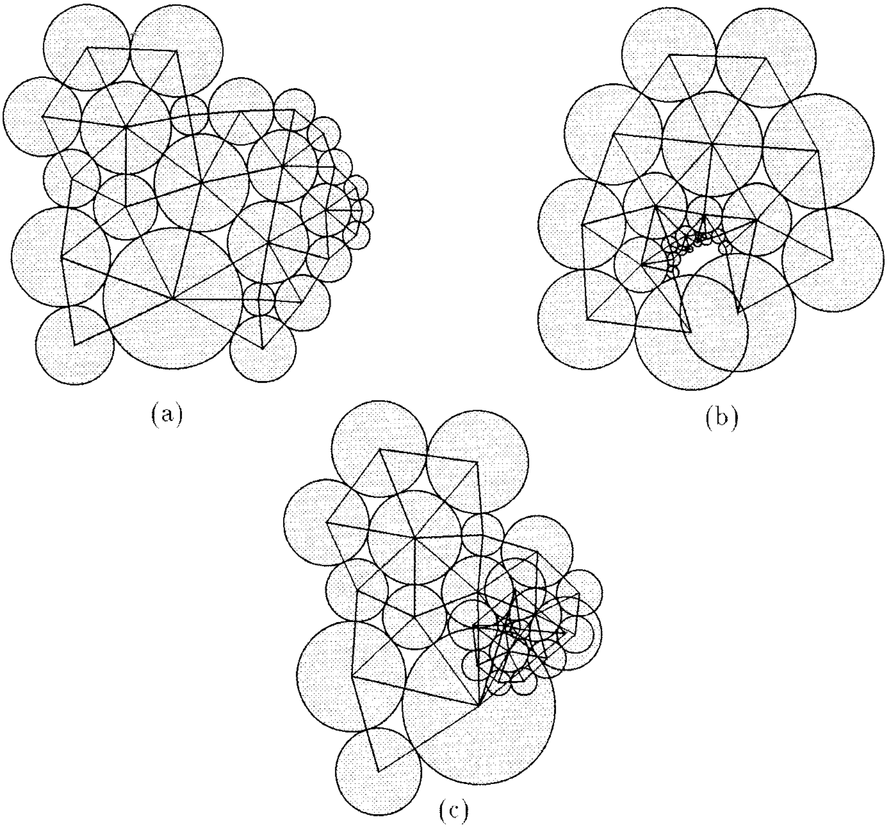

Fig. 3. Different circle packings (and their projected carriers) for the same 2-complex: (a) univalent, (b) locally univalent, and (c) branched.

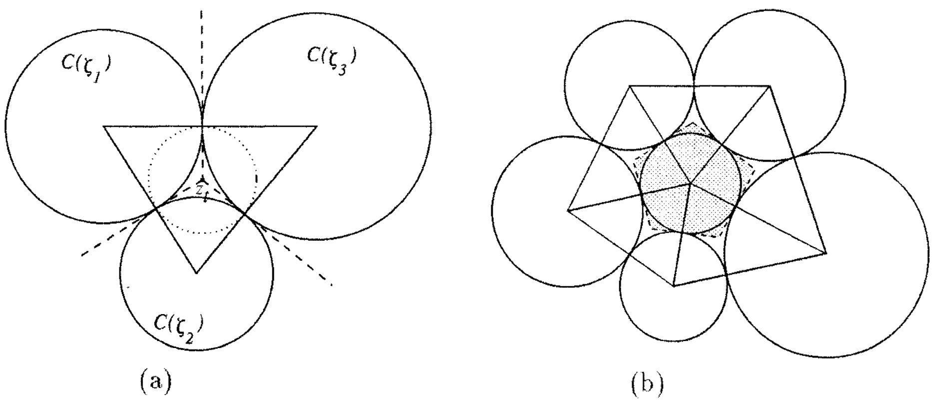

Fig. 4. The point z_t and the volume V_z.

We now address the regularity issues concerning triangulations and volumes generated by circle packings. First, we define the degree $\deg(T)$ of a triangulation T as the least upper bound on the number of edges coming out of any vertex in T. The degree $\deg(\mathcal{P})$ of a packing is then defined by $\deg(\mathcal{P}) := \deg(\mathrm{carr}(\mathcal{P}))$. The key result regarding regularity is the Ring Lemma [RS], which implies the following circle packing regularity.

CP-Regularity. There exists a constant $\kappa = \kappa(d)$, depending only on d, such that for any circle packing $\mathcal{P}$ with $\deg(\mathcal{P}) \le d$,

$$\frac{\mathrm{radius}(\mathcal{C}_{\mathcal{P}})}{\mathrm{radius}(\mathcal{C}'_{\mathcal{P}})} \le \kappa$$

for every pair of adjacent circles $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}'_{\mathcal{P}}$ that are also interior.

In the above, a circle of $\mathcal{P}$ is called interior if its center is an interior vertex of $\mathrm{carr}(\mathcal{P})$, otherwise it is called a boundary circle.

Since the length of an edge in $\mathrm{carr}(\mathcal{P})$ is the sum of radii of two circles centered at the two endpoints of the edge, and since the radius of the inscribed circle of a triangle can be explicitly computed (see [Du3]) from lengths of its edges, the following conclusion is immediately given by CP-regularity.

Corollary 5.1. *For a circle packing $\mathcal{P}$, let $\mathrm{carr}(\mathcal{P}^\circ)$ denote the triangulation obtained from the triangulation $\mathrm{carr}(\mathcal{P})$ by removing all triangles having at least one boundary vertex. Then $\mathrm{carr}(\mathcal{P}^\circ)$ and the associated volume triangulation $\mathrm{carr}(\mathcal{P}^\circ)^*$ have their regularity constants depending only on the degree of $\mathcal{P}$.*

As circle packings can be quite easily generated (see [St3]) once a tangency pattern is given (i.e., a simplicial complex $\mathbb{K}$), the above result shows that triangulations and volumes that are regular can also be delivered. In particular, approximation results from earlier sections can be applied.

We define the quasi-uniformity constant of a circle packing $\mathcal{P}$ as the least upper bound on the ratio $\mathrm{radius}(\mathcal{C}_{\mathcal{P}})/\mathrm{radius}(\mathcal{C}'_{\mathcal{P}})$ for any two circles $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}'_{\mathcal{P}}$ of $\mathcal{P}$. We say that a collection of circle packings $\{\mathcal{P}_n\}$ is regular (respectively, quasi-uniform) if degrees (quasi-uniformity constants) of packings $\mathcal{P}_n$'s are all uniformly bounded. If this is the case, then it follows that the corresponding families of triangulations $\{\mathrm{carr}(\mathcal{P}_n^\circ)\}$ and $\{\mathrm{carr}(\mathcal{P}_n^\circ)^*\}$ are regular (quasi-uniform). From the results of Section 3 and 4 we obtain the following:

Theorem 5.2. *Let Ω be a Jordan domain. Suppose that $\{\mathcal{P}_n\}$ is a collection of circle packings contained in Ω such that*

(1) *radii of circles in $\mathcal{P}_n$ go to 0 as $n \to \infty$,*
(2) *there is a constant $d > 0$ such that $\deg(\mathcal{P}_n) \le d$ for all n, and*
(3) *carriers $\mathrm{carr}(\mathcal{P}_n)$ exhaust Ω.*

Denote by u the solution of the Dirichlet problem $-\triangle u = f$ in Ω and $u = \varphi$ on $\partial\Omega$, where $f \in L^2(\Omega)$ and $\varphi \in C(\partial\Omega)$. Suppose $\bar{\varphi}$ is a continuous extension of φ to some *neighborhood of* $\partial\Omega$. *Write* u_n *for the corresponding discrete solutions for triangulations* $\mathrm{carr}(\mathcal{P}_n^\circ)$ *and volumes* $\mathrm{carr}(\mathcal{P}_n^\circ)^*$. *Then* $\|u - u_n\|_{L^2(\mathrm{carr}(\mathcal{P}_n^\circ))} \to 0$ *as* $n \to \infty$. *Furthermore, if* $\{\mathcal{P}_n\}$ *is a quasi-uniform family, then* $u_n \to u$ *uniformly on compacta of* Ω *as* $n \to \infty$.

Remark 5.3. Notice that "traditional" conditions, such as bounds on angles of triangles in grids [BR], [Ca], [Ci] to ensure the regularity of grids and additional restrictions on these angles [CMM, (3.5)], [Hn, Section 2.3] to ensure the regularity of volumes, are replaced in the above theorem by a single combinatorial condition, i.e., every vertex has no more than d neighbors. (This combinatorial condition is also closely linked with the assumption that circles in packings have disjoint interiors).

We now recall a result about the convergence of discrete Riemann mappings given by circle packings. Suppose that Ω is a Jordan domain. Let $a, b \in \Omega$ be two points. Suppose $\{\mathcal{P}_n\}$ is a collection of circle packings satisfying conditions (1)–(3) of Theorem 5.2. Denote by $\mathbf{D}$ the unit disk in $\mathbf{R}^2$. From the Andreev–Koebe–Thurston theorem [An], [Th1] it follows that for each n there exists a circle packing $\tilde{\mathcal{P}}_n$ contained in $\overline{\mathbf{D}}$, with all boundary circles internally tangent to $\partial\mathbf{D}$, whose carrier is simplicially isomorphic to that of $\mathcal{P}_n$. Moreover, $\tilde{\mathcal{P}}_n$ is normalized so that if a circle in $\mathcal{P}_n$ contains the point a, then the corresponding circle in $\tilde{\mathcal{P}}_n$ is centered at 0, and if a circle in $\mathcal{P}_n$ contains the point b, then the corresponding circle in $\tilde{\mathcal{P}}_n$ is centered in the $(0, 1)$ interval. Let τ_n be a piecewise linear map $\tau_n\colon\ \mathrm{carr}(\mathcal{P}_n) \to \mathbf{D}$ that maps the center of a circle in $\mathcal{P}_n$ to the center of the corresponding circle in $\tilde{\mathcal{P}}_n$ (see Fig. 5). Also, let $\tau_n^\#$ be a piecewise linear map $\tau_n^\#\colon\ \mathrm{carr}(\mathcal{P}_n) \to (0, \infty)$ whose value at the center of a circle in $\mathcal{P}_n$ is the ratio of the radius of the corresponding circle in $\tilde{\mathcal{P}}_n$ to the radius of the circle in $\mathcal{P}_n$. Then we have the following theorem (see [HR], [HS], [RS], [St1], [St2], and [Th2]), where τ' denotes the complex-variable derivative of τ.

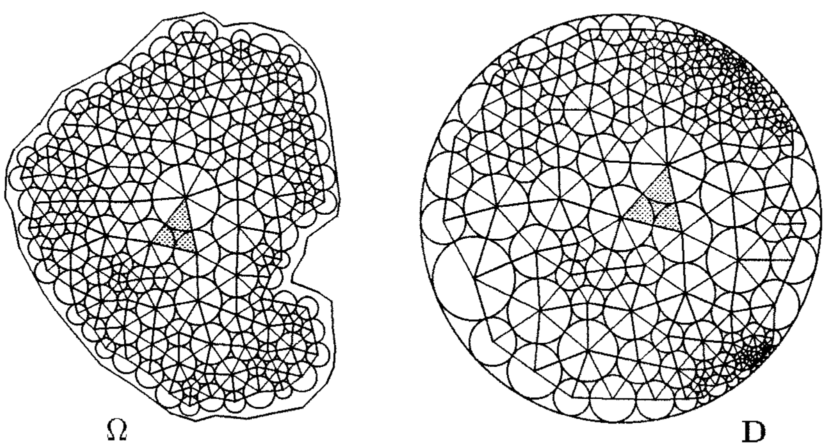

Fig. 5. Two packings giving a discrete Riemann mapping. (Two corresponding triangles are marked for a reference.)

Discrete Riemann Mapping Theorem. *The sequence of maps τ_n converges uniformly on compact subsets of Ω to the Riemann mapping $\tau\colon \Omega \to \mathbf{D}$ with $\tau(a) = 0$ and $\tau(b) > 0$. Moreover, $\tau_n^{\#}$ converge uniformly on compacta of Ω to $|\tau'|$.*

The next observation is a straightforward consequence of the above approximation result and Theorem 5.2.

Corollary 5.4. *Let $\tilde{f} \in L^2(\mathbf{D})$ and $\tilde{\varphi} \in C(\partial\mathbf{D})$. Denote by $\tilde{u}$ the solution of the Dirichlet problem*

$$\begin{cases} -\triangle\tilde{u} = \tilde{f} & \text{in } \mathbf{D}, \\ \tilde{u} = \tilde{\varphi} & \text{on } \partial\mathbf{D}. \end{cases} \qquad \widetilde{(*)}$$

Let Ω be a Jordan domain and let $\tau\colon \Omega \to \mathbf{D}$ be a Riemann mapping. Write u for the solution of the Dirichlet problem $-\triangle u = f$ in Ω and $u = \varphi$ on $\partial\Omega$, where $f(z) := |\tau'(z)|^2 \tilde{f}(\tau(z))$ and $\varphi(z) := \tilde{\varphi}(\tau(z))$. Suppose that $\{\mathcal{P}_n\}$ is a collection of circle packings contained in Ω and satisfying (1)–(3) *of Theorem* 5.2. *Let $\{\tilde{\mathcal{P}}_n\}$ be an associated family of circle packings in $\mathbf{D}$ such that the corresponding maps $\tau_n\colon \operatorname{carr}(\mathcal{P}_n) \to \operatorname{carr}(\tilde{\mathcal{P}}_n)$ and $\tau_n^{\#}$ converge uniformly on compacta of Ω to τ and $|\tau'|$, respectively. Write $\tilde{u}_n$ for the discrete solution of $\widetilde{(*)}$ for triangulations $\operatorname{carr}(\tilde{\mathcal{P}}_n^{\circ})$ and volumes $\operatorname{carr}(\tilde{\mathcal{P}}_n^{\circ})^*$. Then $\|u - \tilde{u}_n \circ \tau_n\|_{L^2(\operatorname{carr}(\mathcal{P}_n^{\circ}))} \to 0$ as $n \to \infty$. Furthermore, if $\{\tilde{\mathcal{P}}_n\}$ is a quasi-uniform family, then $\tilde{u}_n \circ \tau_n \to u$ uniformly on compact subsets of Ω as $n \to \infty$.*

We now move to random walks on circle packings. For details on the subject of random walks in general, the reader should consult, for example, [So] or [Wo]. The notion of random walks on circle packings was introduced in [Du3], and we recall it briefly here. We first define a conductance along an edge. If $\mathcal{P}$ is a circle packing for $\mathbb{K}$ and $\zeta, \zeta' \in \mathbb{K}^0$, $\zeta \sim \zeta'$, then $\int_{\mathcal{P}}(\zeta)\int_{\mathcal{P}}(\zeta')$ is an edge in $\operatorname{carr}(\mathcal{P})$, and the conductance induced by $\mathcal{P}$ along this edge is defined by

$$E_{\mathcal{P}}(\zeta, \zeta') := \frac{|z_t - z_{t'}|}{|\int_{\mathcal{P}}(\zeta) - \int_{\mathcal{P}}(\zeta')|},$$

where t and t' are two triangles in $\operatorname{carr}(\mathcal{P})$ with the common edge $\int_{\mathcal{P}}(\zeta)\int_{\mathcal{P}}(\zeta')$ and, as before, z_t and $z_{t'}$ denote the centers of inscribed circles in triangles t and t', respectively. Then the transition probability from a vertex $\int_{\mathcal{P}}(\zeta)$ to another vertex is defined by

$$Q_{\mathcal{P}}(\zeta, \zeta') := \begin{cases} \dfrac{E_{\mathcal{P}}(\zeta, \zeta')}{\sum_{\zeta'' \sim \zeta} E_{\mathcal{P}}(\zeta, \zeta'')} & \text{for} \quad \zeta \sim \zeta', \\ 0 & \text{for} \quad \zeta \nsim \zeta'. \end{cases}$$

Because $\sum_{\zeta' \sim \zeta} Q_{\mathcal{P}}(\zeta, \zeta') = 1$ for every interior vertex ζ, the matrix $Q_{\mathcal{P}}$ is a probability matrix. We refer to the random walk given by the matrix $Q_{\mathcal{P}}$ as the random walk *induced* by the packing $\mathcal{P}$.

Next, it is standard to introduce the Laplace operator for a random walk by

$$L_{\mathcal{P}} u := (I - Q_{\mathcal{P}})u,$$

where I is the identity matrix and u is a real-valued function defined on the set of vertices of $\mathrm{carr}(\mathcal{P})$. A function u is said to be *harmonic* (with respect to the random walk) if $L_{\mathcal{P}}u = 0$ at every interior vertex. In other words, a function is harmonic if its value at any interior vertex is equal to the weighted average of its values at the neighboring vertices.

Recall that, in Section 3, for a triangulation T we have defined an operator A_T. For $T = \mathrm{carr}(\mathcal{P})$, we write $A_{\mathcal{P}}$ for A_T. By comparing the definitions of operators $L_{\mathcal{P}}$ and $A_{\mathcal{P}}$ we easily get

Proposition 5.5. *Let* $u\colon\ \mathrm{carr}(\mathcal{P})^0 \to \mathbf{R}$. *Then* $L_{\mathcal{P}}u(z) = 0$ *for every interior vertex* z *if and only if* $A_{\mathcal{P}}u(z) = 0$ *for every interior vertex* z. *Furthermore, for any function* $\varphi\colon\ \partial\,\mathrm{carr}(\mathcal{P})^0 \to \mathbf{R}$, *the Dirichlet problem for the random walk induced by* $\mathcal{P}$

$$\begin{cases} L_{\mathcal{P}}u = 0 & \text{in } I\,\mathrm{carr}(\mathcal{P})^0,\\ u = \varphi & \text{on } \partial\,\mathrm{carr}(\mathcal{P})^0,\end{cases}$$

and the discrete Dirchelet problem

$$\begin{cases} A_{\mathcal{P}}u = 0 & \text{in } I\,\mathrm{carr}(\mathcal{P})^0,\\ u = \varphi & \text{on } \partial\,\mathrm{carr}(\mathcal{P})^0,\end{cases}$$

have the same solution.

We apply the approximation results from Section 3 to obtain some information on exit (hitting) probabilities for random walks induced by circle packings. Recall that if X is a subset of the boundary vertices $\partial\,\mathrm{carr}(\mathcal{P})^0$ and $z \in I\,\mathrm{carr}(\mathcal{P})^0$, then the probability $M_{\mathcal{P}}(z, X)$ that the random walk (given by $\mathcal{P}$) starting at z will reach a boundary vertex for the first time and such a vertex will be in X, is called the *exit probability* from z through X. It follows that, for a fixed $z \in I\,\mathrm{carr}(\mathcal{P})^0$, $M_{\mathcal{P}}(z, \cdot)$ is a probability measure on $\partial\,\mathrm{carr}(\mathcal{P})^0$.

A similar notion is available in the continuous case. If Ω is a domain, $z \in \Omega$, and $X \subset \partial\Omega$, then the probability $M(z, X)$ that a Brownian particle starting at z will leave the set Ω for the first time through the set X is called the exit probability from z through X. The next result shows that random walks induced by circle packings mimic the Brownian motion, and that they can be used to estimate exit probabilities of the Brownian motion.

Theorem 5.6. *Let* Ω *be a* C^2 *Jordan domain. Let* γ *be an arc in* $\partial\Omega$. *Suppose* $\{\mathcal{P}_n\}$ *is a quasi-uniform family of circle packings that exhaust* Ω (*i.e.*, (1)–(3) *of Theorem* 5.2 *are satisfied*). *Denote by* γ_n *the set* $\{z \in \partial\mathrm{carr}(\mathcal{P}_n)^0 : z_\partial \in \gamma\}$, *where, as before,* z_∂ *denotes the nearest point on* $\partial\Omega$ *to the point* z. *Then, for any compact subset* ω *of* Ω,

$$\lim_{n\to\infty} \sup_{z\in\omega} |M_{\mathcal{P}_n}(z, \gamma_n) - M(z, \gamma)| = 0.$$

Proof. Let u_n be the solution of the Dirichlet problem: $L_{\mathcal{P}_n}u_n(z) = 0$ for $z \in I\,\mathrm{carr}(\mathcal{P}_n)^0$ and $u_n(z) = 1$ if $z \in \gamma_n$ and $u_n(z) = 0$ if $z \in \partial\,\mathrm{carr}(\mathcal{P}_n)^0 \backslash \gamma_n$. Then $u_n(z) = M_{\mathcal{P}_n}(z, \gamma_n)$ for every $z \in I\,\mathrm{carr}(\mathcal{P}_n)^0$ [DS], [KSK]. Similarly, if u is the solution of the classical Dirichlet problem $\triangle u = 0$ in Ω and $u(z) = 1$ if $z \in \gamma$ and $u(z) = 0$ if $\partial\Omega\backslash\gamma$, then

$u(z) = M(z, \gamma)$ [KS]. Since u_n is also the solution of the corresponding Dirichlet problem for the operator $A_{\mathcal{P}_n}$ by Proposition 5.5, the assertion of the theorem now follows from Example 4.6. □

We conclude this paper with some final remarks.

Remark 5.7. 1. Once again, it should be observed that if we had that the discrete solutions to a Dirichlet problem converge uniformly on compact subsets, regardless of the quasi-uniform condition, then such a condition could be removed from the assumptions in the above theorem.

2. The results of this section can easily be extended to circle packings with overlaps (see [Du3]). Volumes for such circle packings are defined exactly the same as for circle packings without overlaps, that is corners of volumes (i.e., vertices of the dual triangulation) are going to be radical centers of triples of circles. However, volumes will no longer be circumscribed on circles of underlying packings. Nevertheless, by keeping angles of overlaps away from $\pi/2$, a bound on the degree will imply regularity for packings and corresponding volumes. Also, the issue of quasi-uniformity extends without any changes. By allowing for overlaps in packings we add more flexibility to the construction of triangulations and the volumes associated with them.

3. The reader may also be interested in the results [CdVM], [Du2], and [Ma]. As was shown in Section 4(2) of [Du2], the ratio maps for hexagonal triangulations given by solutions of a Dirichlet problem for radius functions of circle packings converge uniformly on compacta to the classical solution of the Dirichlet problem.

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