ON DISCRETE HARMONIC FUNCTIONS

By H. A. HEILBRONN

Received 19 April 1948

1. Introduction

A function $f(x_1, x_2)$ of two real variables x_1, x_2 which are restricted to rational integers will be called discrete harmonic (d.h.) if it satisfies the difference equation

$$4f(x_1,x_2) = f(x_1+1,x_2)+f(x_1-1,x_2)+f(x_1,x_2+1)+f(x_1,x_2-1). \qquad (1\cdot 1)$$

This equation can be considered as the direct analogue either of the differential equation

$$\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2} = 0,$$

or of the integral equation

$$f(x_1,x_2) = \int_0^1 f\{x_1+r\cos(2\pi\theta),\ x_2+r\sin(2\pi\theta)\}\,d\theta$$

in the notation normally employed to harmonic functions.

The object of this paper is to develop the elementary theory of d.h. functions and to investigate how far it corresponds to the theory of harmonic functions. We shall find that many but not all the classical theorems remain valid for d.h. functions.

In many cases the theorems will remain true if we generalize our definition to n dimensions. A function of n (rational integer) variables $x_1, \ldots, x_n$ is called d.h. if

$$2nf(x_1,\ldots,x_n) = f(x_1+1,\ldots,x_n)+\ldots+f(x_1,\ldots,x_n-1). \qquad (1\cdot 2)$$

Several authors have considered similar problems, but none of the results seems to be relevant for our purpose.

2. Definitions and notation

The integer $n \geqslant 2$ always denotes the number of dimensions. We shall write $f(x)$ for $f(x_1, \ldots, x_n)$, denoting by x the point with the coordinates $x_1, \ldots, x_n$. Only points with rational integer coordinates will be considered. Two points x and y will be called neighbouring points if

$$\sum_{\nu=1}^{n}(x_\nu - y_\nu)^2 = 1.$$

A set of points is called connected if any two points of the set can be connected by a chain of neighbouring points which belong to the set. The minimum number of links in the chain will be called the distance of the two points in the set.

A domain consists of two types of points. The interior points which may be any connected set, and the boundary points. These are the points which do not belong to the connected set themselves, but which possess at least one neighbour belonging to the connected set.

It should be noted that a domain is not uniquely determined as a set of points, unless some rule is given to distinguish the interior points of the domain.

A function is d.h. in a domain D if it is defined for all points of D and if (1·2) holds for all interior points x of D. Only real functions will be considered.

A domain will be called finite if it contains only a finite number of points, otherwise it will be called infinite.

We introduce the following abbreviations:

$$o = (0, \ldots, 0),$$

$$u_1 = (1, 0, \ldots, 0), \ \ldots, \quad u_n = (0, 0, \ldots, 1),$$

$$d_\nu f(x) = f(x+u_\nu) - f(x) \quad \text{for} \quad 1 \leqslant \nu \leqslant n,$$

$$\Delta f(x) = \sum_{\nu=1}^{n} (f(x+u_\nu) + f(x-u_\nu) - 2f(x)),$$

$$\sum_D |f'(x)|^2 = \sum_{x,y} (f(x) - f(y))^2,$$

where x and y run through all pairs of neighbours of a finite domain D.

T_R is the domain whose interior points are

$$|x_1| < R, \ldots, |x_R| < R.$$

U_R is the domain whose interior points are

$$|x_1| + \ldots + |x_n| < R.$$

V_R is the domain whose interior points are

$$-R < x_1 \leqslant R, \quad |x_\nu| < R \quad \text{for} \quad 2 \leqslant \nu \leqslant n.$$

For $R > 2$ the domains T_R^* and U_R^* are similarly defined except that the origin is a boundary point.

The constants implied by the symbol O depend on n only.

3. The maximum principle and Dirichlet's principle

Theorem 1. *If $f(x)$ is d.h. on a finite domain D, then $f(x)$ is either a constant or it attains its maximum on D on the boundary only.*

Corollary. *If M is the upper bound of a function $f(x)$ which is d.h., bounded and not constant on an infinite domain D, then $f(x) < M$ for all interior points x of D.*

Proof. Let M be the maximum of $f(x)$ on D, and let x_0 be an interior point of D where $f(x_0) = M$. Then it follows from (1·2) that $f(x) = M$ for all neighbours x of x_0, and by induction that $f(x) = M$ for all points x of D.

Theorem 2. *Let $g(x)$ be a given real function defined on the boundary of a finite domain D. Then there exists one and only one d.h. function $f(x)$ which takes the values $g(x)$ on the boundary of D.*

If $h(x)$ is a real function defined on D which also takes the values $g(x)$ on the boundary of D, then

$$\sum_D |f'|^2 \leqslant \sum_D |h'|^2,$$

and the sign of equality holds only if $f(x) = h(x)$ for all x of D.

Proof. Let $f(x)$ be the function for which $\sum_D |f'|^2$ is a minimum, subject to our boundary condition. Then for each interior point x of D

$$0 = \frac{\partial}{\partial f(x)} \sum_D |f'|^2 = 2\sum_{x'} (f(x) - f(x')),$$

where the sum is extended over all neighbours x' of x; this proves that $f(x)$ is d.h.

If $F(x)$ were another d.h. function satisfying the boundary condition, then $F(x) - f(x) = 0$ by Theorem 1, which shows the uniqueness of $f(x)$ and establishes the theorem.

4. Discrete harmonic polynomials

THEOREM 3. *For every integer $k \geq 1$ there are exactly*

$$\binom{k-2+n}{n-1}\frac{2k+n-1}{k}$$

linearly independent d.h. polynomials of degree not exceeding k.

Proof. For $k = 1$ the theorem is trivial since the $n+1$ polynomials $1, x_1, \ldots, x_n$ are all d.h. Hence we may assume that $k \geq 2$.

An easy count shows that there are

$$\binom{n+k}{n}$$

linearly independent polynomials of degree not exceeding k. Since every polynomial of degree not exceeding $k-2$ can be represented in the form

$$f(x) = \Delta g(x), \quad \deg g(x) \leq k,$$

the operator Δ maps the additive group of polynomials of degree not exceeding k on the subgroup of polynomials of degree not exceeding $k-2$. Since

$$\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x),$$

this mapping is a homomorphism.

Hence the quotient group is isomorphic to the group of all d.h. polynomials of degree not exceeding k; and the latter contains

$$\binom{k+n}{n} - \binom{k-2+n}{n} = \binom{k-2+n}{n-1}\frac{2k+n-1}{k}$$

linearly independent elements.

Examples. For $n = 2$ the d.h. polynomials are

$$1; \quad x_1, x_2; \quad x_1^2 - x_2^2, 2x_1x_2; \quad x_1^3 - 3x_1x_2^2, 3x_1^2x_2 - x_2^3;$$
$$x_1^4 - 6x_1^2x_2^2 + x_2^4 - (x_1^2 + x_2^2), 4x_1^3x_2 - 4x_1x_2^3;$$
$$x_1^5 - 10x_1^3x_2^2 + 5x_1x_2^4 - \tfrac{10}{3}x_1^3, \ldots.$$

We notice that in this sequence each polynomial of degree k is of the form

$$\Re(x_1 + x_2 i)^k + \text{terms of degree not exceeding } k-2,$$
$$\Re(i^{-1}(x_1 + x_2 i)^k) + \text{terms of degree not exceeding } k-2,$$

a fact which is easily verified by direct calculation. It must be pointed out that without further rules the above sequence is not uniquely defined, as there is a considerable degree of freedom in the choice of the lower terms. In this respect the situation differs fundamentally from the continuous case, where the choice of our polynomials is easily made unique by the orthogonality rule.

We also give some examples of the case $n = 3$. By Theorem 3 we have

$$\binom{k+1}{2}\frac{2k+2}{k} = (k+1)^2$$

linearly independent d.h. polynomials of degree not exceeding k, i.e. we have $2k+1$ linearly independent d.h. polynomials of degree k. The first examples are

$$1;\quad x_1, x_2, x_3;\quad x_1x_2, x_2x_3, x_3x_1, x_1^2-x_2^2, x_2^2-x_3^2;$$
$$x_1x_2x_3, x_1^3-3x_1x_2^2, \ldots; x_1x_2(x_1^2+x_2^2-6x_3^2), \ldots,$$
$$x_1^4-6x_1^2x_2^2+x_2^4-(x_1^2+x_2^2), \ldots, x_1x_2(x_1^2-x_2^2), \ldots.$$

The question naturally arises: Is it true that an integer function (i.e. a function d.h. everywhere) can be written as a sum of a unique series of polynomials, at least if $n = 2$? It is difficult to see how the classical theorem on the one-one relation between integer functions and convergent power series can be formulated for d.h. functions, since our sequence of linearly independent polynomials is not uniquely defined. It is trivial that every convergent series of d.h. polynomials converges towards an integer d.h. function. In the opposite direction we start our discussion by proving

THEOREM 4. *If $n = 2$, L is a positive integer, and if $f(x)$ is d.h. on T_L, then we can find a d.h. polynomial $P(x)$ such that $f(x) = P(x)$ on T_L.*

Proof. We make the assertion of the theorem more precise by a further specification of the polynomial $P(x)$. We demand that $P(x)$ shall be a linear combination of polynomials of degree less than $4L-2$ and of the polynomial of degree $4L-2$ of the above sequence which is of the form

$$x_1^{4L-2}-\binom{4L-2}{2}x_1^{4L-4}x_2^2+\ldots.$$

This rule puts $8L-4$ polynomials at our disposal, and the domain T_L has $8L-4$ boundary points. Hence there are two possibilities: Either we can find a linear combination of our $8L-4$ polynomials which assumes the same boundary values as $f(x)$. In this case our theorem is proved. Or the $8L-4$ boundary values of our $8L-4$ polynomials are not linearly independent. In this case there exists a linear combination $Q(x)$ of these polynomials which vanishes on the boundary of T_L, $Q(x)$ not being identically zero. Hence $Q(x)$ vanishes on all points of T_L, and, being d.h., vanishes on all points of U_{2L-1}.

Hence $Q(x)$ has $4L-1$ zeros on the line $x_1 = 0$, and since its degree is less than $4L-1$, $Q(x)$ is either identically zero or divisible by x_1. Similarly, since $Q(x)$ is not identically zero, $Q(x)$ is divisible by x_2. Putting

$$Q(x) = x_1x_2Q_1(x),$$

we have identically

$$Q(x) = x_1x_2Q_1(x) = \alpha\left\{x_1^{4L-2}+\sum_{\nu=1}^{2L-1}(-1)^\nu\binom{4L-2}{2\nu}x_1^{4L-2-2\nu}x_2^{2\nu}\right\}+R(x),$$

where α is a constant and $R(x)$ a polynomial of degree less than $4L-2$. This is only possible if $\alpha = 0$, and if $Q(x)$ is of degree less than $4L-2$; hence $Q_1(x)$ is of degree less than $4L-4$. On each of the four lines $x_1 = \pm 1$, $x_2 = \pm 1$ the polynomial $Q_1(x)$ has at least $4L-4$ zeros, hence it must vanish on these lines identically and

$$Q_1(x_1, x_2) = (x_1^2 - 1)(x_2^2 - 1)\, Q_2(x_1, x_2),$$

where Q_2 is a polynomial of degree less than $4L-8$. Continuing this process we obtain polynomials $Q_l(x_1, x_2)$ of degree less than $4(L-l)$ which satisfy

$$Q_l(x_1, x_2) = (x_1^2 - l^2)(x_2^2 - l^2)\, Q_{l+1}(x_1, x_2)$$

for $1 \leqslant l < L$. Q_{L-1} is of degree less than 4 and has 4 zeros on each of the lines

$$x_1 = \pm(L-1), \quad x_2 = \pm(L-1).$$

Hence Q_{L-1} is identically zero and $Q(x)$ is identically zero, which gives the desired contradiction.

Theorem 4 settles the equation of analytic continuation for a function defined on a square parallel to the axes. It is easily seen that a d.h. function defined on a domain, whose interior points are the lattice points of a convex set in the Euclidean plane, can be continued to a square and therefore is equal to a polynomial.

On the other hand, the function represented by the diagram below (the enclosed points are the interior points of the domain) cannot be continued as a d.h. function to the point marked *; hence it does not equal a d.h. polynomial.

$$\begin{array}{cccccc}
 & 0 & & & & \\
-1 & 0 & 0 & & & \\
4 & 1 & 0 & * & & \\
-1 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 &
\end{array}$$

To sum up: All finite domains can be divided into two classes. Functions d.h. on a domain of the first class can be continued to a square parallel to the axis and can therefore be represented by a d.h. polynomial. To this class belong all convex domains.

Functions d.h. on a domain of the second class cannot always be represented (or even approximated to) by d.h. polynomials. To this class belong the domain in the above diagram and all domains not 'simply connected'.

For infinite domains there is a similar division into two classes; but functions d.h. on an infinite domain of the first class can only be continued to integer functions which can be approximated to by a sequence of polynomials. Naturally this approximation is not uniform, and no analogue to the absolute convergence in the classical theory seems to exist.

5. LIOUVILLE'S THEOREM

We next proceed to prove the analogue of Liouville's theorem on bounded integer functions.

THEOREM 5. *If $f(x)$ is d.h. everywhere and satisfies the inequality*

$$|f(x)| \leqslant M \tag{5·1}$$

for all x, where M is a constant, then $f(x)$ is constant.

Proof (indirect). We may assume without loss of generality that

$$d_1 f(x) = g(x)$$

is not zero everywhere. Since $|g(x)| \leqslant 2M$, there exists a positive upper bound m such that

$$|g(x)| \leqslant m \leqslant 2M \tag{5·2}$$

everywhere, and without loss of generality

$$g(x) > m - \epsilon$$

somewhere for every $\epsilon > 0$.

We choose an integer l such that $lm > 2M$,

and a positive δ such that $lm(1-(2n)^l \delta) > 2M$.

Then we can find a point $x^{(0)}$ such that

$$g(x^{(0)}) > m - \delta. \tag{5·3}$$

We put $x^{(\lambda)} = x^{(0)} + \lambda u_1$ for $0 < \lambda \leqslant l$.

Then, since $g(x)$ is d.h., $2ng(x^{(0)}) \leqslant (2n-1)\,m + g(x^{(1)})$,

and by (5·3) $g(x^{(1)}) \geqslant m - 2n\delta$.

Applying the same argument again, we obtain by induction for $0 \leqslant \lambda \leqslant l$

$$g(x^{(\lambda)}) \geqslant m - (2n)^\lambda \delta.$$

Hence

$$2M \geqslant f(x^{(l)}) - f(x^{(0)}) = \sum_{\lambda=0}^{l-1} g(x^{(\lambda)}) \geqslant \sum_{\lambda=0}^{l-1} (m - (2n)^\lambda \delta)$$
$$> l(m - (2n)^l \delta) > 2M,$$

which is the desired contradiction.

A natural extension of Theorem 5 is

THEOREM 6. *If $f(x)$ is d.h. everywhere and satisfies the inequality*

$$f(x) = O(1 + (|x_1| + \ldots + |x_n|)^k)$$

everywhere, where k is an integer, then $f(x)$ is a polynomial of degree not exceeding k.

A proof of this theorem will be given in the last paragraph.

6. Some special boundary problems

The simplest problem in two dimensions refers to the domain whose interior points are all points except the origin. We prove the following generalization of Theorem 5:

THEOREM 7. *If $n = 2$, and if $f(x)$ is bounded for all x and d.h. for all $x \neq o$, then $f(x)$ is a constant.*

Proof. We may assume without loss of generality that

$$f(o) = 1, \quad 0 \leqslant f(x) \leqslant 2. \tag{6·1}$$

Let $R > 1$ be an integer and let $f_R(x)$ be the function d.h. on U_R^* which satisfies

$$f(o) = 1, \quad f(x) = 0 \quad \text{for} \quad |x_1| + |x_2| = R.$$

Clearly $0 \leqslant f_R(x) \leqslant f_{R+1}(x) \leqslant 1, \quad f_R(x) \leqslant f(x)$

for all points of U_R^*, hence the sequence $f_1(x), f_2(x), f_3(x), \ldots$ converges towards a limit $f_\infty(x)$ which satisfies $0 \leqslant f_R(x) \leqslant f_\infty(x) \leqslant 1, \quad f_\infty(x) \leqslant f(x),$

provided that x belongs to U_R^*. If we can show that $f_\infty(x) = 1$ everywhere, it will follow that $f(x) \geqslant 1$ everywhere, and since (6·1) is symmetric in $f(x)$ and $2-f(x)$, that $2-f(x) \geqslant 1$, or $f(x) \leqslant 1$.

We define
$$g_R(x) = 1 - \frac{\log(1+|x_1|+|x_2|)}{\log(1+R)}.$$

Then
$$\sum_{U_R^*} |g_R'(x)|^2 = O\sum_{\nu=1}^{R} \nu\left(\frac{\log(1+\nu)}{\log(1+R)} - \frac{\log \nu}{\log(1+R)}\right)^2$$
$$= O\left(\sum_{\nu=1}^{R} \nu^{-1}\log^{-2} R\right) = O(\log^{-1} R).$$

Since $f_R(x)$ is d.h. on U_R^*, and since $f_R(x)$ has the same boundary values as $g_R(x)$, we have, by Theorem 2,
$$\sum_{U_R^*} |f_R'(x)|^2 \leqslant \sum_{U_R^*} |g_R'(x)|^2 = O(\log^{-1} R).$$

Hence for each x, as $R \to \infty$
$$d_1 f_R(x) = o(1), \quad d_2 f_R(x) = o(1),$$
$$d_1 f_\infty(x) = d_2 f_\infty(x) = 0, \quad f_\infty(x) = 1.$$

In three or more dimensions the situation is different. We shall prove only

Theorem 8. *For $n = 3$ there exists a function which is bounded everywhere, d.h. everywhere except at the origin and not a constant.*

Before we proceed to prove this theorem we shall establish a lemma which is well known in the classical calculus of variations.

Lemma 1. *Let $u(\xi_1, \xi_2, \xi_3)$ be continuous for $1 \leqslant \xi_1^2+\xi_2^2+\xi_3^2 \leqslant (2R)^2$ and let u have continuous bounded partial first derivatives almost everywhere in this domain. Let*
$$u(\xi_1,\xi_2,\xi_3) \geqslant \tfrac{1}{2} \quad \textit{for} \quad \xi_1^2+\xi_2^2+\xi_3^2 = 1$$
and
$$u(\xi_1,\xi_2,\xi_3) = 0 \quad \textit{for} \quad \xi_1^2+\xi_2^2+\xi_3^2 = (2R)^2.$$
Then
$$\iiint\left\{\left(\frac{\partial u}{\partial \xi_1}\right)^2 + \left(\frac{\partial u}{\partial \xi_2}\right)^2 + \left(\frac{\partial u}{\partial \xi_3}\right)^2\right\} d\xi_1 d\xi_2 d\xi_3 \geqslant \pi \frac{2R}{2R-1}, \quad 1 \leqslant \xi_1^2+\xi_2^2+\xi_3^2 \leqslant (2R)^2.$$

Proof. On introducing polar coordinates ρ, θ, λ the integral is transformed into
$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos\theta\, d\theta \int_{-\pi}^{\pi} d\lambda \int_1^{2R} \rho^2\left\{\left(\frac{\partial u}{\partial \rho}\right)^2 + \rho^2\left(\frac{\partial u}{\partial \theta}\right)^2 + \rho^2\cos^2\theta\left(\frac{\partial u}{\partial \lambda}\right)^2\right\} d\rho$$
$$\geqslant \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos\theta\, d\theta \int_{-\pi}^{\pi} d\lambda \int_1^{2R} \rho^2\left(\frac{\partial u}{\partial \rho}\right)^2 d\rho.$$

Since, by the Cauchy-Schwarz inequality
$$\frac{2R-1}{2R}\int_1^{2R} \rho^2\left(\frac{\partial u}{\partial \rho}\right)^2 d\rho = \int_1^{2R} \rho^{-2} d\rho \int_1^{2R} \rho^2\left(\frac{\partial u}{\partial \rho}\right)^2 d\rho \geqslant \left(\int_1^{2R} \frac{\partial u}{\partial \rho} d\rho\right)^2 \geqslant \tfrac{1}{4},$$
the result follows.

Proof of Theorem 8. As in the proof of Theorem 7 we define $f_R(x)$ as the function d.h. on U_R^* which satisfies $f_R(o) = 1$ and vanishes at all other points of the boundary of U_R^*. $f_R(x)$ increases with R and tends to a limit $f_\infty(x)$. We want to show that $f_\infty(x) = 1$ is not true everywhere, hence we may assume at once that for sufficiently large R
$$f_R(\pm u_\nu) \geqslant \tfrac{1}{2} \quad \text{for} \quad 1 \leqslant \nu \leqslant 3.$$

We now proceed to construct a continuous function $u_R(\xi_1, \xi_2, \xi_3)$ which will coincide with $f_R(x)$ if $\xi_1 = x_1, \xi_2 = x_2, \xi_3 = x_3$. For every cube parallel to the axes, of unit volume, whose vertices have integer coordinates, we can find 8 constants $\alpha_{000}, \alpha_{100}, \ldots, \alpha_{111}$ such that the trilinear form

$$u_R(\xi_1, \xi_2, \xi_3) = \alpha_{000} + \alpha_{100}\xi_1 + \alpha_{010}\xi_2 + \alpha_{001}\xi_3 + \alpha_{110}\xi_1\xi_2 + \alpha_{101}\xi_1\xi_3 + \alpha_{011}\xi_2\xi_3 + \alpha_{111}\xi_1\xi_2\xi_3$$

takes the same values at the vertices of the cube as the function $f_R(x)$ ($f_R(x)$ being zero outside U_R^*).

This function $u_R(\xi_1, \xi_2, \xi_3)$ is uniquely defined everywhere and satisfies the assumptions of Lemma 1. Further for each of our cubes C

$$\iiint_C \left\{\left(\frac{\partial u_R}{\partial \xi_1}\right)^2 + \left(\frac{\partial u_R}{\partial \xi_2}\right)^2 + \left(\frac{\partial u_R}{\partial \xi_3}\right)^2\right\} d\xi_1 d\xi_2 d\xi_3 \leqslant \sum_C |f_R'(x)|^2.$$

Hence, by Lemma 1, since every edge belongs to four different cubes

$$4\sum_{U_R^*} |f_R'(x)|^2 \geqslant \pi \frac{2R}{2R-1},$$

and, putting $A_R = \sum_{U_R^*} |f_R'(x)|^2$, for $R > 1$, we obtain

$$A_R \geqslant \tfrac{1}{4}\pi.$$

Let σ_R be defined by $\qquad \sigma_R = f_R(\pm u_\nu) \quad (1 \leqslant \nu \leqslant 3).$

We now define a function $g_R(x, \rho)$ by

$$g_R(o, \rho) = 1, \quad g_R(x, \rho) = \rho f_R(x) \quad \text{for} \quad x \neq o, \ x \text{ in } U_R^*,$$

where ρ is a positive variable. Then

$$\sum_{U_R^*} |g_R'(x, \rho)|^2 = 6(1-\rho\sigma_R)^2 + \rho^2\{A_R - 6(1-\sigma_R)^2\}.$$

As $g_R(x, 1) = f(x)$ and as $f_R(x)$ is d.h., this expression must be a minimum if $\rho = 1$. Differentiation with respect to ρ gives

$$\frac{d}{d\rho}\sum_{U_R^*} |g_R'(x, \rho)|^2 = -12\sigma_R(1-\rho\sigma_R) + 2\rho\{A_R - 6(1-\sigma_R)^2\}.$$

If we put $\rho = 1$ we obtain

$$0 = 12\sigma_R + 2A_R - 12, \quad \sigma_R = 1 - \tfrac{1}{6}A_R \leqslant 1 - \frac{\pi}{24},$$

whence $\qquad f_\infty(\pm u_\nu) \leqslant 1 - \dfrac{\pi}{24} \quad (1 \leqslant \nu \leqslant 3).$

We conclude this section with the construction of a function which has many useful properties.

THEOREM 9. *For $n \geqslant 2$ there exists a function $h(x)$ d.h. everywhere with the following properties:*

$$h(o) = 1, \tag{6·2}$$

$$h(x) = 0 \quad \textit{for} \quad x_n = 0, \ x \neq o, \tag{6·3}$$

$$h(x) > 0 \quad \textit{for} \quad x_n > 0, \tag{6·4}$$

$$h(x) = O(x_n^{1-n}) \quad \textit{for} \quad x_n > 0. \tag{6·5}$$

Proof. Let $\zeta_1, \ldots, \zeta_{n-1}$ be $n-1$ real continuous variables. We define $\phi(\zeta_1, \ldots, \zeta_{n-1})$ as the smaller root of the quadratic equation

$$\phi + \phi^{-1} + 2 \sum_{\nu=1}^{n-1} \cos \zeta_\nu = 2n. \tag{6·6}$$

An elementary calculation shows easily that

$$(4n-2)^{-1} < \phi(\zeta_1, \ldots, \zeta_{n-1}) \leqslant 1.$$

We put

$$h(x) = (2\pi)^{1-n} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \cos(x_1 \zeta_1) \ldots \cos(x_{n-1}\zeta_{n-1}) \, \phi^{x_n}(\zeta_1, \ldots, \zeta_{n-1}) \, d\zeta_1 \ldots d\zeta_{n-1}.$$

Since ϕ satisfies (6·6) we see at once that $h(x)$ is d.h. everywhere. (6·2) and (6·3) are now trivial.

To prove (6·5) we observe that

$$\phi(\zeta_1, \ldots, \zeta_n) = 1 - (\zeta_1^2 + \ldots + \zeta_{n-1}^2)^{\frac{1}{2}} + O(\zeta_1^2 + \ldots + \zeta_n^2).$$

Hence
$$\underset{\zeta_1^2 + \ldots + \zeta_{n-1}^2 \leqslant x_n^{-1}}{\int \cdots\cdots \int} \phi^{x_n}(\zeta_1, \ldots, \zeta_{n-1}) \, d\zeta_1 \ldots d\zeta_{n-1}$$
$$= O \underset{\zeta_1^2 + \ldots + \zeta_{n-1}^2 \leqslant x_n^{-1}}{\int \cdots\cdots \int} \exp[-x_n(\zeta_1^2 + \ldots + \zeta_{n-1}^2)^{\frac{1}{2}}] \, d\zeta_1 \ldots d\zeta_{n-1}$$
$$= O \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp[-x_n(\zeta_1^2 + \ldots + \zeta_{n-1}^2)^{\frac{1}{2}}] \, d\zeta_1 \ldots d\zeta_{n-1}$$
$$= O(x_n^{1-n}),$$

whereas
$$\underset{\zeta_1^2 + \ldots + \zeta_{n-1}^2 \geqslant x_n^{-1}}{\int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi}} \phi^{x_n}(\zeta_1, \ldots, \zeta_{n-1}) \, d\zeta_1 \ldots d\zeta_{n-1}$$
$$= O \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \{\exp[-x_n^{-\frac{1}{2}} + O(x_n^{-1})]\}^{x_n} \, d\zeta_1 \ldots d\zeta_{n-1}$$
$$= O[\exp(-x_n^{\frac{1}{2}})] = O(x_n^{1-n}).$$

This proves (6·5), since $\qquad h(x) \leqslant h(o + x_n u_n).$

Let $M(u)$ denote the minimum or the greatest lower bound of $h(x)$ for $x_n = u$, $u \geqslant 0$. Clearly
$$2M(u) \geqslant M(u+1) + M(u-1) \quad \text{for} \quad u > 0.$$
Since $M(0) = 0$ and $\lim_{u \to \infty} M(u) = 0$, it follows that $M(u) \geqslant 0$. This proves $h(x) \geqslant 0$ for $x_n \geqslant 0$, and (6·4) follows by Theorem 1.

7. GENERAL BOUNDARY PROBLEMS

THEOREM 10. *Let S be an infinite domain with at least one boundary point. If $F(x)$ is a bounded function defined for all boundary points of S, then the definition of $F(x)$ can be extended to all points of S such that $F(x)$ is bounded and d.h. in S.*

If $n = 2$ this process is unique; if $n > 2$ it is not in general unique.

Proof. Without loss of generality we may assume that

$$0 \leqslant F(x) \leqslant 1$$

on the boundary of S. Let x_0 be an interior point of S, and let S_1 be the domain which contains only x_0 as interior point. The domains $S_2, S_3, \ldots$ are defined by induction by the following rule: The interior points of S_{l+1} are *all* points of S_l which are interior points of S. Clearly each S_l contains all preceding domains and every x which is an interior point of S belongs to the interior of S_l, if l is sufficiently large. For each $l > 0$ we find the function $f_l(x)$ d.h. on S_l which vanishes on the boundary points of S_l which are interior to S, and which satisfies $f_l(x) = F(x)$ on the boundary points of S_l which are boundary points of S. Clearly $f_l(x)$ is an increasing sequence for fixed x, hence it converges to a function $F(x)$ for all x in S, which has the required properties.

That $F(x)$ need not be unique for $n > 2$ has been demonstrated by Theorem 8.

For $n = 2$ we assume that $|F(x)| \leqslant M$ for all points of S and $F(x) = 0$ on the boundary of S, and we have to show that $F(x)$ vanishes on all points of S. We may assume that o is a boundary point of S. Let W_R for integers $R > 1$ be the largest domains with o as boundary point whose interior points are interior points of U_R and of S. It is clear that W_R exists and is unique for sufficiently large R, and that every boundary point of W_R is either a boundary point of U_R or of S. The function

$$F(x) + M\{1 - f_R(x)\}$$

is d.h. on all interior points x of W_R, $f_R(x)$ having the same meaning as in the proof of Theorem 7. If x lies on the boundary of S, $F(x) = 0$, whereas $f_R(x) = 0$ if x lies on the boundary of U_R. Hence

$$F(x) + M\{1 - f_R(x)\} \geqslant 0$$

for all x of W_R. Since each fixed interior point x of S belongs to W_R if R is sufficiently large, and since $\lim_{R\to\infty} f_R(x) = 1$, it follows that $F(x) \geqslant 0$ for all x of S. In a similar way one proves $F(x) \leqslant 0$ which completes the proof of the theorem.

THEOREM 11. *Let S be an infinite domain with at least one boundary point. If $F(x)$ is a function defined for all boundary points of S, then the definition of $F(x)$ can be extended to all points of S such that $F(x)$ is d.h. in S.*

Proof. For each positive integer m we define $F_m(x)$ on the boundary points of S by

$$F_m(x) = \begin{cases} m & \text{if} \quad F(x) > m, \\ F(x) & \text{if} \quad |F(x)| \leqslant m, \\ -m & \text{if} \quad F(x) < -m. \end{cases}$$

Then the sequence $F_m(x)$ converges towards $F(x)$ on the boundary of S and we can define $F_m(x)$ as a d.h. function in S by virtue of Theorem 10. But it does not follow that the sequence converges for all points of S.

Let $x_1, x_2, x_3, \ldots$ run through all interior points of S in any order. We shall construct for each integer $l \geqslant 0$ an infinite sequence $f_{l,m}(x)$ of functions d.h. on S which converges towards $F(x)$ as $m \to \infty$ and for which the limit

$$\lim_{m\to\infty} f_{l,m}(x_\nu) = F(x_\nu) \quad (1 \leqslant \nu \leqslant l)$$

exists and is independent of l.

https://doi.org/10.1017/S0305004100024713 Published online by Cambridge University Press

For $l = 0$, the sequence $f_{0,m}(x) = F_m(x)$ has the required property. We apply induction and assume that we have constructed sequences $f_{0,m}(x), \ldots, f_{l,m}(x)$ according to our rule. Then the sequence $f_{l,m}(x_{l+1})$ has either a convergent subsequence or it has a subsequence such that the quotient of two consecutive terms tends to infinity. In the first case our proposition can obviously be established for $l+1$. In the second case let $g_{l,m}(x)$ be the subsequence of $f_{l,m}(x)$ with

$$g_{l,m+1}(x_{l+1})/g_{l,m}(x_{l+1}) \to \infty \quad \text{as} \quad m \to \infty.$$

Put
$$\rho_m = \frac{g_{l,m}(x_{l+1})}{g_{l,m}(x_{l+1}) - g_{l,m+1}(x_{l+1})}.$$

Then
$$\rho_m \to 0,$$

and the sequence
$$f_{l+1,m}(x) = (1-\rho_m)\, g_{l,m}(x) + \rho_m g_{l,m+1}(x)$$

has the required property, since

$$\lim_{m\to\infty} f_{l+1,m}(x) = \lim_{m\to\infty} g_{l,m}(x)$$

for all x for which the limit on the right exists, and since

$$f_{l+1,m}(x_{l+1}) = 0 \quad \text{for} \quad m = 1, 2, 3, \ldots.$$

Thus our function $F(x)$ is defined and d.h. everywhere in S.

It is easily seen that in Theorem 11 the function $F(x)$ is never unique. The answer to the question how many linearly independent functions $F(x)$ will satisfy Theorem 11 depends in a rather complicated way on the structure of the domain S. Even the question whether the number of linearly independent solutions is finite or infinite is not easily answered.

A further problem arises if we subject $F(x)$ to some conditions restricting its magnitude. We have seen that if $n = 2$ the boundedness of the solution implies uniqueness, and for certain types of domain a weaker restriction will still preserve uniqueness. This leads to theorems of the Phragmén-Lindelöf type. Many interesting questions arise, but it seems hopeless to formulate a theorem of reasonable generality.

8. Some elementary inequalities

It is easily seen that an analogue of Poisson's formula can be established for d.h. functions. We limit ourselves to a rectangular domain D, whose interior points are given by the inequalities
$$\alpha_\nu < x_\nu < \beta_\nu \quad (\nu = 1, \ldots, n).$$

If $f(x)$ is d.h. on D, and if y is a point of D we have

$$f(y) = \sum_x K(x,y)\, f(x),$$

where x runs through all boundary points of D. Here $K(x,y)$ is the d.h. function of y which vanishes on all boundary points of D except at the point $y = x$ where it assumes the value $K(x,x) = 1$. It follows at once that for every interior point y of D

$$0 < K(x,y) < 1.$$

The exact calculation of $K(x,y)$ is very tedious even for relatively small domains D. We shall restrict ourselves to prove

https://doi.org/10.1017/S0305004100024713 Published online by Cambridge University Press

THEOREM 12. $$K(x,y) = O(|x-y|^{1-n}),$$
where $|x-y|$ is the Euclidean distance from x to y.

Proof. Without loss of generality we may assume that $x = o$, $y_n \geq n^{-1}|y-o|$. Then for all y on the boundary of D $$0 \leq K(x,y) \leq h(y).$$
Since this inequality is also true for all points y interior to D, the theorem follows from (6·5).

As an easy consequence of Theorem 12 we prove

THEOREM 13. *If $f(x)$ is d.h. on a rectangular domain D, and if y is an interior point of D which has a Euclidean distance not less than $R > 0$ from every boundary point of D, then*
$$f^2(y) = O(R^{1-n}) \sum_x f^2(x),$$
where x runs through all boundary points of D.

Proof. Since $$1 = \sum_x K(x,y),$$
Theorem 12 gives $$f^2(y) = \left\{\sum_x K(x,y)\, f(x)\right\}^2 \leq \sum_x K^2(x,y) \sum_x f^2(x)$$
$$\leq \max_x K(x,y) \sum_x f^2(x) = O(R^{1-n}) \sum_x f^2(x).$$

For the proof of our last theorem we require

LEMMA 2. *If $f(x)$ is d.h. on V_R and if*
$$f(x) \geq 0 \quad \textit{for} \quad x_1 > 0, \qquad f(x) \leq 0 \quad \textit{for} \quad x_1 \leq 0 \tag{8·1}$$
on the boundary of V_R, then $d_1 f(o) \geq 0$.

If $f(x)$ also satisfies $$f(x) + f(u_1 - x) = 0 \tag{8·2}$$
on all points of V_R, then (8·1) *holds on all points of V_R.*

Proof. Put $$g(x) = \begin{cases} f(x) & \text{if} \quad (x_1 - \tfrac{1}{2})\, f(x) \geq 0, \\ 0 & \text{if} \quad (x_1 - \tfrac{1}{2})\, f(x) < 0. \end{cases}$$
Then $g(x) = f(x)$ on the boundary of V_R and, if (8·2) holds,
$$\sum_{V_R} |g'(x)|^2 \leq \sum_{V_R} |f'(x)|^2.$$
Since $f(x)$ in d.h. on V_R, it follows that $g(x) = f(x)$ for all points of V_R and (8·1) holds for all points of V_R. This proves the second part of the lemma.

To prove the first part, we put
$$F(x) = f(x) - f(u_1 - x).$$
Then $F(x)$ satisfies (8·2) and (8·1) for all x of V_R; hence
$$d_1 f(o) = F(u_1) \geq 0.$$

THEOREM 14. *If $f(x)$ is d.h. on V_R and if on V_R*
$$|f(x)| \leq M,$$
then $$d_1 f(o) = O(MR^{-1}).$$

Proof. Without loss of generality $M = 1$. Let $f_R(x)$ be the function d.h. on V_R which has the boundary values $$f_R(x) = \begin{cases} 1 & \text{if} \quad x_1 > 0, \\ -1 & \text{if} \quad x_1 \leq 0. \end{cases}$$

Clearly $f_R(x)$ satisfies (8·2) and (8·1) on all points of V_R. The function

$$g_R(x) = f_R(x) - \frac{x_1 - \frac{1}{2}}{R + \frac{1}{2}}$$

is also d.h. on V_R and satisfies (8·2) and (8·1) there. In particular,

$$\begin{aligned} g_R(x) &\geqslant 0 \quad \text{for} \quad x_1 = R, \\ g_R(x) &\leqslant 0 \quad \text{for} \quad x_1 = -R+1, \\ g_R(x) &= 0 \quad \text{for} \quad x_1 = R+1 \text{ and } x_1 = -R. \end{aligned}$$

Therefore we have on the boundary of T_R

$$d_1 g_R(x) \begin{cases} = 2 - \dfrac{1}{R+\frac{1}{2}} & \text{for} \quad x_1 = 0, \\ \leqslant 0 & \text{for} \quad |x_1| = R, \\ = -\dfrac{1}{R+\frac{1}{2}} & \text{for} \quad 0 < |x_1| < R. \end{cases}$$

Hence, by Theorem 12, $$d_1 g_R(o) \leqslant O(R^{-1}),$$

and, by Lemma 2, $$d_1 f(o) \leqslant d_1 f_R(o) = \frac{1}{R+\frac{1}{2}} + d_1 g_R(o) \leqslant O(R^{-1}).$$

In a similar way it is proved that

$$d_1 f(o) \geqslant O(R^{-1})$$

and the theorem is established.

With the help of Theorem 14, Theorem 6 follows at once in the usual way.

THE ROYAL FORT
BRISTOL 8