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ON THE BEHAVIOR OF SOME CELLULAR AUTOMATA RELATED TO BOOTSTRAP PERCOLATION[1]

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We consider some deterministic cellular automata on the state space $\{0,1\}^{\mathbb{Z}^d}$ evolving in discrete time, starting from product measures. Basic features of the dynamics include: 1's do not change, translation invariance, attractiveness and nearest neighbor interaction. The class of models which is studied generalizes the bootstrap percolation rules, in which a 0 changes to a 1 when it has at least l neighbors which are 1. Our main concern is with critical phenomena occurring with these models. In particular, we define two critical points: p_c, the threshold of the initial density for convergence to total occupancy, and π_c, the threshold for this convergence to occur exponentially fast. We locate these critical points for all the bootstrap percolation models, showing that they are both 0 when $l \le d$ and both 1 when $l > d$. For certain rules in which the orientation is important, we show that $0 < p_c = \pi_c < 1$, by relating these systems to oriented site percolation. Finally, these oriented models are used to obtain an estimate for a critical exponent of these models.

1. Introduction. The fields of interacting particle systems, mathematical statistical mechanics and percolation have benefited very much from their interrelations. Here we study a family of models which has arisen in the interface among these areas.

Cellular automata, such as those studied in this article, may be considered as interacting particle systems [see Liggett (1985) for a survey of this field]. The relations between the models that we consider and percolation will become clear in many of the proofs given, but can already be guessed from the fact that some of these systems are known as "bootstrap percolation." Finally, relations with statistical mechanics, while not so explicit in this article, were clearly present, for instance, in the article by Chalupa, Leath and Reich (1979), where bootstrap percolation was introduced in connection to disordered magnetic systems. Also in Aizenman and Lebowitz (1988) the motivation for studying these systems came from the (nonequilibrium statistical mechanics) problem of metastability.

Our main concern in this article will be with the critical behavior of our models, that is, how their behavior changes qualitatively as some parameters cross certain values (critical points). As usual, one of the main tools in the analysis of such phenomena will be a sort of renormalization procedure by

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which we compare systems with different values of the parameters when we modify the scales of length and time. In our case, due to the simplicity of the models, this scheme will be accomplished in a very straightforward fashion, so that our examples may give an idea of these techniques even to a reader who is not familiar with this approach.

The article will be essentially self-contained, but for motivations and reviews on cellular automata we refer the reader to Griffeath (1988), Toffoli and Margolus (1987), Vichniac (1984), Wolfram (1983, 1986). In the next section we introduce the models and the problems and explain how the rest of the article is organized.

2. The models and problems. The models considered in this article are defined on the lattice $\mathbb{Z}^d$, where $\mathbb{Z}$ is the set of integers and $d = 1, 2, \dots$ is the space dimensionality. The systems evolve in discrete time $t = 0, 1, 2, \dots$. To each element (site) of $\mathbb{Z}^d$, x, we associate at each instant of time t a random variable $\eta_t(x)$, which can assume the values 0 and 1. We say that the site x is empty (resp. occupied) at time t if $\eta_t(x) = 0$ (resp. 1). $\eta_t \in \{0,1\}^{\mathbb{Z}^d}$ will represent the function that associates $x \in \mathbb{Z}^d$ to $\eta_t(x)$. Elements of $\{0,1\}^{\mathbb{Z}^d}$ are called configurations. The system will be always started at $t = 0$, from a translation invariant product random field; that is, the random variables $\eta_0(x)$, $x \in \mathbb{Z}^d$ are i.i.d. with $P(\eta_0(x) = 0) = q$, $P(\eta_0(x) = 1) = p = 1 - q$. $p \in [0,1]$ is called the initial density. The system evolves then according to the following sort of deterministic rules:

1. If $\eta_t(x) = 1$, then $\eta_{t+1}(x) = 1$ (1's are stable).
2. If $\eta_t(x) = 0$ and η_t belongs to a certain set $\mathscr{C}_x$ (the sets $\mathscr{C}_x$, $x \in \mathbb{Z}^d$, specify the model), then $\eta_{t+1}(x) = 1$; otherwise $\eta_{t+1}(x) = 0$.

In this article the sets $\mathscr{C}_x$ will always obey several restrictions. (a) *Translation invariance.* We define $\theta_x\eta$ by $(\theta_x\eta)(y) = \eta(y - x)$ and we assume that $\mathscr{C}_x = \{\eta\colon \theta_{-x}\eta \in \mathscr{C}_0\}$. In particular the set $\mathscr{C}_0 =: \mathscr{C}$ specifies the model. (b) *Nearest neighbor interaction.* We define $\mathscr{N}_x = \{y \in \mathbb{Z}^d\colon \|x - y\| = 1\}$, where $\|\cdot\|$ is the l_1 norm on $\mathbb{Z}^d$ ($\|x\| = |x_1| + \cdots + |x_d|$). We assume that if $\eta \in \mathscr{C}_x$ and $\eta(y) = \eta'(y)$ for every $y \in \mathscr{N}_x$, then $\eta' \in \mathscr{C}_x$. Informally, each site is influenced only by its nearest neighbors at each step of the evolution. (c) *Attractiveness.* We define on $\{0,1\}^{\mathbb{Z}^d}$ the partial order given by $\eta \le \eta'$ if $\eta(x) \le \eta'(x)$ for every $x \in \mathbb{Z}^d$. We assume that if $\eta \in \mathscr{C}_x$ and $\eta \le \eta'$, then $\eta' \in \mathscr{C}_x$. Informally, the more 1's we have at time t, the more 1's we will have at time $t + 1$.

The set $\mathscr{C}$ may be specified by a set $\mathscr{D}$ of subsets of $\mathscr{N} := \mathscr{N}_0$ via

$$\mathscr{D} = \{A \subset \mathscr{N}\colon \eta(x) = 1 \text{ for all } x \in A \Rightarrow \eta \in \mathscr{C}\}.$$

Observe that, by attractiveness, if $A \in \mathscr{D}$ and $A \subset B$, then $B \in \mathscr{D}$.

In order to give some examples we define the elements of $\mathbb{Z}^d$, $e_1 = (1,0,0,\dots,0), \dots, e_d = (0,0,0,\dots,1)$. $|A|$ will denote the cardinality of the set A.

EXAMPLES.

1. *Bootstrap percolation.* Take $l \in \{0, \dots, 2d\}$ and set
$$\mathscr{D} = \{A \subset \mathscr{N}: |A| \geq l\}.$$
A 0 becomes a 1 if at least l of its neighbors are 1's.
2. *The basic model.* This is the particular case of bootstrap percolation with $l = d$. In $d = 2$ this is the model studied by van Enter (1987) and Aizenman and Lebowitz (1988).
3. *The modified basic model.*
$$\mathscr{D} = \{A \subset \mathscr{N}: A \cap \{-e_i, +e_i\} \neq \varnothing \text{ for } i = 1, \dots, d\}.$$
In this model a 0 becomes a 1 if in each one of the d coordinate directions it has at least one neighbor which is a 1.
4. *Oriented models.* Take $(a_1, \dots, a_d) \in \{-1, +1\}^d$. For each one of these 2^d choices we have one of the oriented models defined by
$$\mathscr{D} = \{A \subset \mathscr{N}: \{a_1 e_1, a_2 e_2, \dots, a_d e_d\} \subset A\}.$$
In case $a_i = +1$, for $i = 1, \dots, d$, we call the model the *basic oriented model*.

Given two models defined, respectively, by $\mathscr{D}_1$ and $\mathscr{D}_2$, we say that the latter dominates the former if $\mathscr{D}_1 \subset \mathscr{D}_2$. Informally, if a 0 becomes a 1 in the former, the same occurs in the latter. The following statements are clearly true. The bootstrap percolation model with $l = l_1$ dominates the one with $l = l_2$ if $l_1 \leq l_2$. The basic model dominates the modified basic model and this one dominates all the oriented models.

On $\{0, 1\}$ we take the discrete topology and on $\{0, 1\}^{\mathbb{Z}^d}$ and $\{0, 1\}^{\mathbb{Z}^d \times \{0, 1, \dots\}}$ the corresponding product topologies and Borel σ algebras, Σ and $\overline{\Sigma}$. Once the dimension d, the set $\mathscr{C}$ and the initial density p are specified, we denote by $P_p(\cdot)$ the probability measure on $(\{0, 1\}^{\mathbb{Z}^d \times \{0, 1, \dots\}}, \overline{\Sigma})$ corresponding to the process $(\eta_t: t \geq 0)$.

Let $\mathscr{M}$ be the set of probability measures on $(\{0, 1\}^{\mathbb{Z}^d}, \Sigma)$. On $\mathscr{M}$ we define the following partial order. If $\mu, \nu \in \mathscr{M}$, we say that ν dominates μ and write $\mu \leq \nu$ if
$$\int f(\eta)\, d\mu(\eta) \leq \int f(\eta)\, d\nu(\eta)$$
for every continuous nondecreasing function $f: \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ (the corresponding partial order on $\{0, 1\}^{\mathbb{Z}^d}$ is the one defined before). For the properties of this notion of partial order see Section 2 of Chapter 2 of Liggett (1985).

Let μ_0^p = (translation invariant product measure with density p) be the initial distribution on $\{0, 1\}^{\mathbb{Z}^d}$, and μ_t^p be the corresponding distribution at time t. Since the 1's are stable we have
$$\mu_0^p \leq \mu_1^p \leq \mu_2^p \leq \cdots .$$
Since $\{0, 1\}^{\mathbb{Z}^d}$ and consequently $\mathscr{M}$ are compact, it follows that μ_t^p converges weakly to a probability distribution $\mu^p \in \mathscr{M}$, which depends on p. μ^p is clearly

translation invariant. The asymptotic density is defined as

$$\rho(p) = \mu^p\{\eta: \eta(0) = 1\}.$$

From attractiveness we have also that $p_1 \le p_2$ implies $\mu^{p_1} \le \mu^{p_2}$. In particular $\rho(p_1) \le \rho(p_2)$.

Now we list various questions which can be raised about these systems.

1. For which values of p does the system "fill all the space"? More precisely, when is it the case that $\rho(p) = 1$? Clearly $\rho(0) = 0$ and $\rho(1) = 1$, and from the monotonicity of $\rho(\cdot)$, it is natural to define

   $$p_c = \inf\{p \in [0,1]: \rho(p) = 1\}.$$

   What can be said about p_c?
2. If the system fills the whole space, does it do it exponentially fast? To be more precise, we define the random time

   $$T = \inf\{t \ge 0: \eta_t(0) = 1\}$$

   and ask whether there are $\gamma, C \in (0,\infty)$ such that

   $$P_p(T > t) \le Ce^{-\gamma t}.$$

   We can define for every p

   $$\gamma(p) = \sup\left\{\gamma \ge 0: \text{there exists a } C < \infty \text{ such that } P_p(T > t) \le Ce^{-\gamma t}\right\}.$$

   $\gamma(\cdot)$ is clearly a monotonic nondecreasing function. It is natural then to define another critical point

   $$\pi_c = \inf\{p \in [0,1]: \gamma(p) > 0\}.$$

   Clearly $p_c \le \pi_c$, Since $\rho(p) = 1 - \lim_{t\to\infty} P_p(T > t)$. What else can be said about π_c?
3. What is the behavior of the characteristic quantities $\rho(p)$ and $\gamma(p)$ near the critical points p_c and π_c? Is it the case that $\lim_{p\searrow\pi_c}\gamma(p) = 0$? And if this is true, is there a corresponding critical exponent $\nu \in (0,\infty)$ such that $\gamma(p) \sim (p - \pi_c)^\nu$ as $p \searrow \pi_c$? Is it the case that $P_{\pi_c}(T > t) \sim t^{-k}$ for a critical exponent $\kappa \in (0,\infty)$ as $t \to \infty$? If ν and κ indeed exist, how do they depend on the dimension? The preceding symbol $\sim$ may either have the strong meaning $f(x) \sim x^\alpha$ iff $\lim[f(x)/x^\alpha] \in (0,\infty)$ or the weak meaning $f(x) \sim x^\alpha$ iff $\lim(\log f(x)/\log x) = \alpha$.
4. Is it always the case that $p_c = \pi_c$? Similar questions have been answered affirmatively for percolation by Menshikov (1986), Menshikov, Molchanov and Sidorenko (1986) and Aizenman and Barsky (1987), and for ferromagnetic Ising models by Aizenman, Barsky and Fernandez (1987).
5. When $0 < \rho(p) < 1$, μ^p has a nontrivial structure. It is easy to show that μ^p is translation invariant and ergodic w.r.t. translations. Using the methods of Section 2 of Chapter 3 of Liggett (1985) one can show that μ^p has positive correlations. Is it the case that these correlations decay exponentially fast? And how fast does μ_t^p converge to μ^p?

We obtained some partial answers to these questions. Now we summarize our results and explain how the rest of the article is organized. In Section 3 we will show that for the modified basic model in every dimension d, $p_c = \pi_c = 0$ (and hence the same is true for the bootstrap percolation models with $l \le d$). This extends results by van Enter (1987), who showed that for bootstrap percolation with $l = 2$, $p_c = 0$ in every dimension. In Section 4 we will show that the oriented models are closely related to oriented site percolation and obtain as a consequence that for these models $0 < p_c = \pi_c < 1$. In Section 5 we obtain results by comparing the systems with the oriented models. If a model does not dominate any oriented model (as is the case for the bootstrap percolation models with $l > d$), then it is easy to show that $p_c = \pi_c = 1$. On the other hand, for models that dominate some oriented model, $p_c \le \pi_c < 1$ and we will show that for these models $P_{\pi_c}(T > t) \ge Ct^{-d+1}$ if $t > 1$, for some strictly positive constant C. In particular, $\gamma(\pi_c) = 0$, and if the critical exponent κ exists, then $\kappa \le d - 1$.

We make two remarks about notation. The symbols C, C_1, C_2, C', $C(\varepsilon, N), \dots$ will always denote strictly positive finite constants, whose exact value is irrelevant and may even change from line to line. In Section 3, where we use an induction argument on the dimension, we keep the dimension explicit in the notation. But in the other sections, where the dimension is generic but is kept fixed, we omit it in the notation.

3. The modified basic model. In $d = 1$ the modified basic model is trivial. Clearly

$$\begin{aligned} P_p(T > t) &= P_p(\eta_0(x) = 0 \text{ for } x = -t, \dots, t) \\ &= q^{2t+1} = q \exp(-(2\log(1/q))t). \end{aligned}$$

Hence $p_c = \pi_c = 0$ and $\gamma(p) = 2\log(1/q)$. $\nu = 1$ and $\kappa = 0$, since for every $t \ge 0$,

$$P_{\pi_c}(T > t) = 1.$$

In $d = 2$ van Enter (1987) proved that $p_c = 0$ (his proof for the basic model applies to the modified basic model). We will prove the following theorem by induction on the dimension.

THEOREM 3.1. *In all dimensions, for the modified basic model* $p_c = \pi_c = 0$.

The proof of this theorem will be broken into several propositions and lemmas, some of which are interesting in their own right. In this section we will use notations which make the dimension explicit as $p_c(d)$, $\pi_c(d)$, T^d and $\gamma_d(p)$. First we define various subsets of $\mathbb{Z}^d$:

$$Q_k^d = \{x \in \mathbb{Z}^d : |x_i| \le k, i = 1, \dots, d\}.$$

Given $r \in \{1, \dots, d\}$, $1 \le i_1 < i_2 < \cdots < i_r \le d$ and $a_s \in \{-1, +1\}$ for each

$s = 1, \dots, r$, we define

$$L^d_{((i_1,\dots,i_r),(a_1,\dots,a_r))}(k)$$
$$= \left\{x \in \mathbb{Z}^d \colon x_{i_s} = a_s k \text{ for } s = 1,\dots,r \text{ and } |x_j| < k \text{ for } j \notin \{i_1,\dots,i_r\}\right\}.$$

For each fixed r, let $\mathscr{I}_r$ be the set of possible indices above; that is,

$$\mathscr{I}_r = \{((i_1,\dots,i_r),(a_1,\dots,a_r)) \colon 1 \le i_1 < \cdots < i_r \le d \text{ and } a_s \in \{-1,+1\} \text{ for each } s = 1,\dots,r\}.$$

Set also

$$\mathscr{I} = \bigcup_{r=1}^{d} \mathscr{I}_r.$$

Observe that for $I \in \mathscr{I}_r$, $L^d_I(k)$ is a $(d-r)$-dimensional hypercube of side $2k-1$. [In case $r = d$, $L^d_I(k)$ is a point.] Also, the collection $\{L^d_I(k) \colon I \in \mathscr{I}\}$ forms a partition of $Q^d_k \setminus Q^d_{k-1}$. In particular

$$Q^d_k = Q^d_{k-1} \cup \Bigl(\bigcup_{I \in \mathscr{I}} L^d_I(k)\Bigr).$$

We will define now various dynamics related to the modified basic model. In each of them 1's will never change and 0's may change to 1's according to the state of the neighboring sites. The d-dimensional dynamics restricted to a set $\Gamma \subset \mathbb{Z}^d$ is obtained by freezing the states of the sites outside of Γ as 0 and letting the system inside of Γ evolve as the modified basic model. As in Aizenman and Lebowitz (1988), we will say that a finite set $\Gamma \subset \mathbb{Z}^d$ is internally spanned by the configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ if starting from the configuration η^Γ defined by

$$\eta^\Gamma(x) = \begin{cases} \eta(x), & \text{if } x \in \Gamma, \\ 0, & \text{if } x \notin \Gamma, \end{cases} \tag{3.1}$$

and letting the system evolve according to the d-dimensional dynamics restricted to Γ, Γ will eventually become completely occupied. Set

$$R^d(N,P) = P(Q^d_N \text{ is internally spanned by a random configuration chosen according to a product measure with density } p).$$

For every $I \in \mathscr{I}_r$ and k, the $(d-r)$-dimensional dynamics restricted to $L^d_I(k)$ is obtained by freezing the states of the sites outside of $L^d_I(k)$ as 0 and letting the system inside of $L^d_I(k)$ evolve like a $(d-r)$-dimensional modified basic model. More precisely, inside of $L^d_I(k)$ a 0 changes to 1 when it has at least one occupied neighbor in each one of the $(d-r)$ different directions defined by the vectors $\pm e_j$, $j \notin \{i_1,\dots,i_r\}$, where $I = ((i_1,\dots,i_r),(a_1,\dots,a_r))$. We say that the set $L^d_I(k)$ is internally spanned by the $(d-r)$-dimensional dynamics by a configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ if, starting from $\eta^{L^d_I(k)}$ and letting the system evolve according to the $(d-r)$-dimensional dynamics restricted to $L^d_I(k)$, $L^d_I(k)$ will eventually become completely occupied. Observe that

$$P(L_I^d(k) \text{ is internally spanned by the } (d-r)\text{-dimensional dynamics by a random configuration chosen according to a product measure with density } p) = R^{d-r}(k-1, p). \tag{3.2}$$

LEMMA 3.1. *If Q_{k-1}^d is completely occupied in the configuration η and for each $r \in \{1, \dots, d-1\}$ and $I \in \mathscr{I}_r$, $L_I^d(k)$ is internally spanned by the $(d-r)$-dimensional dynamics by η, then Q_k^d is internally spanned by η.*

PROOF. First we have to observe another fact about the geometry of the sets $L_I^d(k)$. If $x \in L_I^d(k)$ for some $I = ((i), (a)) \in \mathscr{I}_1$, then the site $x - ae_i$ is in Q_{k-1}^d; and if $x \in L_I^d(k)$ for some $I = ((i_1, \dots, i_r), (a_1, \dots, a_r)) \in \mathscr{I}_r$ for some $r \in \{2, \dots, d\}$, then all the sites of the form $x - a_{i_s} e_{i_s}$, $s = 1, \dots, r$, are in $\bigcup_{J \in \mathscr{I}_{r-1}} L_J^d(k)$.

From the first observation above and the hypothesis of the lemma, it follows that each time that a 0 becomes a 1 in the $(d-1)$-dimensional dynamics restricted to some $L_I^d(k)$, $I \in \mathscr{I}_1$, the same occurs with respect to the d-dimensional dynamics restricted to Q_k^d. Hence $\bigcup_{I \in \mathscr{I}_1} L_I^d(k)$ will eventually become completely occupied in this latter dynamics, at a random time Θ_1. At the time Θ_1, the configurations in the regions $L_I^d(k)$, $I \in \mathscr{I}_2$, obviously dominate the configurations in these regions at time 0, so that these sets will be internally spanned by the $(d-r)$-dimensional dynamics by the configuration at time Θ_1. As above, by the second remark in the first paragraph of this proof, $\bigcup_{I \in \mathscr{I}_2} L_I^d(k)$ will then become completely occupied by the d-dimensional dynamics restricted to Q_k^d at a random time Θ_2. Proceeding by induction in the same fashion as before, one can show that $\bigcup_{r=1}^{d-1} (\bigcup_{I \in \mathscr{I}_r} L_I^d(k))$ will become completely occupied by the d-dimensional dynamics restricted to Q_k^d at an almost surely finite random time Θ_{d-1}. But then it is easy to see that at time $\Theta_{d-1} + 1$, Q_k^d will become completely occupied by the same dynamics, completing the proof. □

PROPOSITION 3.1. *For the modified basic model, $\gamma_d(p) \le \gamma_{d-1}(p)$ and $\pi_c(d) \ge \pi_c(d-1)$.*

PROOF. Consider the $(d-1)$-dimensional space $\{0\} \times \mathbb{Z}^{d-1} \subset \mathbb{Z}^d$. As before, consider the $(d-1)$-dimensional dynamics restricted to this space, in which a 0 becomes a 1 when it has at least one occupied neighbor in each one of the $(d-1)$ directions different from the one defined by $\pm e_1$. If a 0 changes to 1 by the action of the d-dimensional dynamics at a site of $\{0\} \times \mathbb{Z}^{d-1}$, then the same change occurs under the $(d-1)$-dimensional dynamics. So for each t,

$$P_p(T^{d-1} > t) \le P_p(T^d > t),$$

from which the proposition follows immediately. □

PROPOSITION 3.2. *For the modified basic model,*

$$1 - R^d(N, p) \le C(p)\exp\left(-\frac{\gamma_d(p)}{4^d}N\right),$$

where $C(p) < \infty$ may depend on the dimension.

PROOF. The proposition is easily verified in $d = 1$. We suppose now that it holds in dimensions $1, 2, \dots, d - 1$ and will show that it holds in dimension d. If $\gamma_d(p) = 0$, there is nothing to be proven, so we suppose also that $\gamma_d(p) > 0$. From Proposition 3.1, then,

$$\gamma_1(p) \ge \gamma_2(p) \ge \cdots \ge \gamma_d(p) > 0. \tag{3.3}$$

Set $M = \lfloor N/4 \rfloor$ and let F_N be the event that for every $k > M$ and $r = 1, \dots, d - 1$ each one of the sets $L_I^d(k)$, $I \in \mathscr{I}_r$, is internally spanned by the $(d - r)$-dimensional dynamics when the system starts from a random configuration chosen according to a product measure with density p. From (3.2), (3.3) and the induction hypothesis,

$$P\big((F_N)^c\big) \le \sum_{r=1}^{d-1} C_r(p)\exp\left(-\frac{\gamma_{d-r}(p)}{4^r}\frac{N}{4}\right). \tag{3.4}$$

Let G_N be the event that Q_M^d becomes completely occupied by the d-dimensional dynamics restricted to Q_N^d when the system starts from a random configuration chosen according to a product measure with density p. Now we use the fact that since the interaction is among nearest neighbors, "the effects travel with a maximum speed 1." To be more precise, define for $x \in \mathbb{Z}^d$,

$$r(x) = \min\{n : x \in Q_n^d\}.$$

Now, for $x \in Q_N^d$, if x is vacant at time $t \le N - r(x)$ in the d-dimensional dynamics restricted to Q_N^d and started from $\eta^{Q_N^d}$, then x is also vacant at this time t if the system is started from η and evolves according to the dynamics of the modified basic model. [This can be proven easily by induction on the value of $N - r(x)$.] Using translation invariance, we obtain

$$\begin{aligned} P\big((G_N)^c\big) &\le P_p\Big(\bigcup_{x \in Q_M^d} \{\eta_{N-M}(x) = 0\}\Big) \\ &\le |Q_M^d| \cdot P_p\big(\eta_{N-M}(0) = 0\big) \\ &\le C_0(p)\exp\left(-\frac{\gamma_d(p)}{2}N\right). \end{aligned} \tag{3.5}$$

Now from Lemma 3.1, it follows that

$$1 - R^d(N, p) \le P\big((F_N)^c\big) + P\big((G_N)^c\big). \tag{3.6}$$

Equations (3.3)–(3.6) show that the proposition will hold then in dimension d, if it holds in all smaller dimensions. □

LEMMA 3.2. *For the modified basic model, if $\pi_c(d-1) = 0$, then for every $p > 0$,*

$$\lim_{N\to\infty} R^d(N,p) = 1.$$

PROOF. We say that the origin is a good site in the configuration η, if $\eta(0) = 1$ and for every $k = 1, 2, \ldots,$ $r = 1, \ldots, d-1$ and $I \in \mathscr{I}_r$, the set $L_I^d(k)$ is internally spanned by the $(d-r)$-dimensional dynamics by the configuration η. We say that the site x is a good site in the configuration η, if the origin is a good site in the shifted configuration $\theta_{-x}\eta$ defined by $(\theta_{-x}\eta)(y) = \eta(y+x)$. From (3.2) we have

$$\begin{aligned}\alpha(p) &:= P(\text{the origin is a good site in a configuration } \eta \\ &\qquad \text{chosen randomly according to a product} \\ &\qquad \text{measure with density } p) \\ &= p\prod_{k=1}^{\infty}\prod_{r=1}^{d-1}(R^r(k,p))^{C(d,r)},\end{aligned} \tag{3.7}$$

where $C(d,r)$ is the cardinality of $\mathscr{I}_r$ in dimension d. But from Propositions 3.1, 3.2 and the hypothesis $\pi_c(d-1) = 0$, it follows that for every $p > 0$ and $r = 1, \ldots, d-1$, $1 - R^r(k,p)$ goes to 0 exponentially fast as k goes to infinity. Hence (3.7) implies that

$$\alpha(p) > 0. \tag{3.8}$$

Let F_N and G_N be defined as in the proof of Proposition 3.2. Then it follows from Lemma 3.1 and the fact that $M \le N/4$ that

$$\begin{aligned}P(G_N) \ge P(&\text{there is a good site inside of } Q_M^d \text{ in a configuration } \eta \\ &\text{chosen randomly according to a product} \\ &\text{measure with density } p).\end{aligned} \tag{3.9}$$

From (3.8), (3.9) and ergodicity,

$$\lim_{N\to\infty} P(G_N) = 1. \tag{3.10}$$

From the hypothesis and Proposition 3.1, (3.4) holds and implies that

$$\lim_{N\to\infty} P(F_N) = 1. \tag{3.11}$$

The proposition follows from (3.6), (3.10) and (3.11). □

Define now the "correlation length"

$$\xi_d(p) = \inf\{N: R^d(N,p) \ge 1 - 1/(2(2d-1))\}.$$

PROPOSITION 3.3. *For the modified basic model,*

$$\gamma_d(p) \ge C_d/\xi_d(p),$$

where C_d is a positive constant which does not depend on p.

PROOF. We suppose that $\xi_d(p) < \infty$, since otherwise the proposition is trivial. The proposition will be proved using a renormalization procedure. First we show that for large enough p, there are $\gamma(p), C(p) \in (0, \infty)$, such that

$$P_p(T^d > t) \le C(p) e^{-\gamma(p)t}. \tag{3.12}$$

For this we use a relation between the modified basic model and site percolation [see Kesten (1982) or Grimmett (1989) for more on percolation]. Let $\mathscr{C}_t(0)$ be the vacant cluster of the origin at time t; that is,

$$\mathscr{C}_t(0) = \Big\{x \in \mathbb{Z}^d\text{: for some } n \text{ there are } 0 = x_0, x_1, \dots, x_n = x \text{ such that}$$
$$\|x_i - x_{i+1}\| = 1,\ i = 0, \dots, n-1 \text{ and } \eta_t(x_j) = 0 \text{ for } j = 1, \dots, n\Big\}.$$

Set

$$D_k = \{x \in \mathbb{Z}^d\colon \|x\| = k\}$$

and

$$R_t(0) = \sup\{k\colon \mathscr{C}_t(0) \cap D_k \ne \varnothing\},$$

with the convention $\sup \varnothing = -\infty$. $R_t(0)$ is the radius of $\mathscr{C}_t(0)$. It is easy to see that $R_{t+1}(0) \le R_t(0) - 1$. Indeed, if $R_t(0) = k$, then all the sites in $\mathscr{C}_t(0) \cap D_k$ have d occupied neighbors, one in each direction, and hence they will become occupied at time $t + 1$. So

$$P_p(T^d > t) \le P_p(R_0(0) \ge t). \tag{3.13}$$

For $p > 1 - 1/(2d - 1)$ we can use now a simple Peierls type of estimate. Since the number of self-avoiding walks with length l starting from the origin is not larger than $(2d-1)^{l-1} \cdot (2d)$, for $t > 1$,

$$P_p(R_0(0) \ge t) \le \sum_{l=t}^{\infty} q^{l+1}(2d-1)^{l-1}(2d) \le Ce^{-\gamma t}. \tag{3.14}$$

Equation (3.12) follows when $p > 1 - 1/(2d-1)$ from (3.13) and (3.14).

The renormalization procedure will be introduced now. It is essentially the same renormalization scheme introduced by Aizenman and Lebowitz (1988) [see their derivation of relation (4.4) in that article], except that here we are also concerned with the time scale. Recall the definition of $Q_N^d = \{-N, \dots, N\}^d$, and consider its translates

$$Q_{N,k}^d = \{x \in \mathbb{Z}^d\colon x - k(2N+1) \in Q_N^d\}, \qquad k \in \mathbb{Z}^d.$$

These hypercubes form a partition of $\mathbb{Z}^d$. Each one of them is thought of as a site of the renormalized lattice. We say that a site k of the renormalized lattice is occupied if in the original lattice all the sites in $Q_{N,k}^d$ are occupied. The state of occupancy of the renormalized sites evolves in a way that in the proper time scale dominates the evolution of the modified basic model. To see this, observe that if at time t the renormalized site k has at least one occupied renormalized neighbor in each one of the d different directions, then at time $t + 2dN + 1$ this renormalized site will be occupied. This is not hard to prove, and a formal argument will be given in a more general setting in Lemma 5.1. We define the renormalized time τ by

$$t = (2N + 1)^d + (2dN + 1)\tau. \tag{3.15}$$

The motivation for the shift $(2N + 1)^d$ is the following. To know whether a region $\Gamma \subset \mathbb{Z}^d$ is internally spanned by a configuration η it is enough to wait $|\Gamma|$ units of time. This is so because if after $|\Gamma|$ units of time τ is not internally spanned, then there is a time $t < |\Gamma|$ so that from time t to $t + 1$ no change occurred in the configuration and then the final configuration has been reached. Using attractiveness we see now that if in the original lattice we start from the product measure with density p, then at the renormalized time $\tau = 0$ [i.e., $t = (2N + 1)^d$], the distribution of the state of occupancy of the renormalized sites dominates the product measure with density $R^d(N, p)$; at the renormalized times $\tau = 1, 2, \dots,$ this distribution will dominate the evolution of this product measure under the dynamics of the modified basic model in τ units of time. In conclusion,

$$P_p(T^d > t) \le P_{R^d(N,p)}(T^d > \tau). \tag{3.16}$$

But choosing $N = \xi_d(p)$ we have, for every $p > 0$, $R^d(N, p) \ge 1 - (1/2(2d - 1)) =: p_0$, and from (3.16), (3.12) and (3.15),

$$\begin{aligned} P_p(T^d > t) &\le C(p_0)\exp\left(-\gamma(p_0) \cdot \left\lfloor \left(t - (2N + 1)^d/(2dN + 1)\right)\right\rfloor\right) \\ &\le C'(p)\exp(-(C_d/\xi_d(p)) \cdot t). \end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 3.1. From Lemma 3.2, if $\pi_c(d - 1) = 0$, then $\xi_d(p) < \infty$ for every $p > 0$. Now from Proposition 3.3, $\gamma_d(p) > 0$ for every $p > 0$ and, hence, $\pi_c(d) = 0$. The theorem follows then from the fact that $0 \le p_c(d) \le \pi_c(d)$ and that $\pi_c(1) = 0$, as remarked in the beginning of this section. □

From Theorem 1(ii) in Aizenman and Lebowitz (1988), we have that there are $0 < C_1 \le C_2 < \infty$ such that

$$e^{C_1/p} \le \xi_2(p) \le e^{C_2/p}.$$

So $\xi_2(p)$ diverges much faster than a power of $1/p$. For higher dimensions, this theorem in Aizenman and Lebowitz (1988), combined with the arguments used to prove Proposition 3.1, imply that for a constant $C_d \in (0, \infty)$,

$$\xi_d(p) \ge e^{C_d/p}.$$

Our results and methods are nevertheless compatible with a much faster divergence of $\xi_d(p)$ as $p \to 0$.

PROBLEM 3.1. In $d \ge 3$, how fast does $\xi_d(p)$ diverge as $p \to 0$?

This is a relevant question in part because in simulations, as those quoted in the following, a certain asymptotic form for the relation between $\xi_d(p)$ and p is usually assumed (finite size scaling).

From Proposition 3.1 and the computations in $d = 1$ it follows that in all dimensions

$$\gamma_d(p) \le 2\log(1/q).$$

In particular, $\lim_{p\to 0}\gamma_d(p) = 0$, and if the critical exponent ν exists, then $\nu \ge 1$ in all dimensions. In the opposite direction we only proved in Proposition 3.3

$$\gamma_d(p) \ge C_d/\xi_d(p),$$

which is compatible with a very fast convergence of $\gamma_d(p)$ to 0.

PROBLEM 3.2. How fast does $\gamma_d(p) \to 0$ as $p \to 0$?

After this article was finished, Enrique Andjel seems to have given a partial answer to this question in the two-dimensional case, showing that if ν exists, then $\nu \le 2$.

Since the basic model dominates the modified basic model, it is clear that also for the former, in any dimension, $p_c = \pi_c = 0$. Therefore, our rigorous result contradicts predictions made for the basic model in three dimensions based on simulations. Kogut and Leath (1981) simulated the system on finite cubes, the largest one having side 44, and concluded that there was a discontinuous [in $\rho(p)$] transition at $p_c \in [0.091, 0.104]$. Adler and Aharony (1988) made simulations on cubes of sides up to 50 and concluded that $p_c \in [0.094, 0.114]$. Manna, Stauffer and Heermann (1989) simulated the system on cubes of sides between 32 and 704 and found indications that $p_c \in [0.059, 0.069]$.

4. The oriented models. In this section we will explore the strong relation which exists between the oriented models and oriented site percolation. The dimension is arbitrary and will not appear in the notation in this or the next section.

Oriented site percolation is defined on the lattice $\mathbb{Z}^d$. We say that $(x_1, x_2, \dots, x_n)$ is an oriented path in $\mathbb{Z}^d$ if $x_i \in \mathbb{Z}^d$, $i = 1, \dots, n$ and $n = 1$, or $x_{i+1} - x_i \in \{e_1, \dots, e_d\}$, $i = 1, \dots, n-1$. Given a random field $\{\alpha(x)\colon x \in \mathbb{Z}^d\}$, where $\alpha(x) \in \{0,1\}$, we say that x is occupied (resp. vacant) w.r.t. α if $\alpha(x) = 1$ (resp. 0). The oriented occupied cluster of the site x wrt α is the random set

$$C^o_\alpha(x) = \{y \in \mathbb{Z}^d\colon \text{there is an oriented path } (x_1, \dots, x_n) \text{ such that } x = x_1, y = x_n \text{ and } \alpha(x_i) = 1 \text{ for } i = 1, \dots, n\}.$$

Observe that if $\alpha(x) = 0$, then $C^o_\alpha(x) = \varnothing$. The oriented vacant cluster of the site x w.r.t. α, $C^v_\alpha(x)$, is defined analogously, with the condition $\alpha(x_i) = 0$ replacing the condition $\alpha(x_i) = 1$. The range of $C^u_\alpha(x)$, $u = o, v$, is defined as 0 if $C_\alpha^u(x) = \varnothing$ and otherwise as

$$A_\alpha^u(x) = \sup\left\{1 + \sum_{i=1}^{d} |y_i - x_i| : y \in C_\alpha^u(x)\right\}.$$

We say that oriented percolation of occupied (resp. vacant) sites occurs in α if $A_\alpha^o(0) = \infty$ [resp. $A_\alpha^v(0) = \infty$].

Let β_p be a product random field on $\mathbb{Z}^d$ such that $P(\beta_p(x) = 1) = p = 1 - P(\beta_p(x) = 0)$ for every $x \in \mathbb{Z}^d$. Now we define

$$\theta^u(p) = P\big(A_{\beta_p}^u(0) = \infty\big), \qquad u = o, v.$$

Clearly $\theta^v(p) = \theta^o(1-p)$. $\theta^o(p)$ [resp. $\theta^v(p)$] is a monotonic nondecreasing (resp. nonincreasing) function of p. We define

$$p_c^o = \inf\{p \in [0,1]: \theta^o(p) > 0\}$$

and

$$p_c^v = \sup\{p \in [0,1]: \theta^v(p) > 0\}.$$

Then $p_c^v = 1 - p_c^o$. It is well known that in $d = 1$, $p_c^o = 1$, $p_c^v = 0$, but in $d \geq 2$, $0 < p_c^o < 1$ and $0 < p_c^v < 1$.

In the next proposition we consider the basic oriented model, but it is clear that an analogous statement holds for other oriented models. For the basic oriented model set

$$T(x) = \inf\{t \geq 0: \eta_t(x) = 1\}.$$

Proposition 4.1. *For the basic oriented model*,

$$T(x) = A_{\eta_0}^v(x).$$

Proof. Consider first the case $A_{\eta_0}^v(x) = \infty$. In this case there is an infinite sequence $x = x_1, x_2, \dots$ of sites of $\mathbb{Z}^d$ such that for $i = 1, 2, \dots,$ $x_{i+1} - x_i \in \{e_1, \dots, e_d\}$ and $\eta_0(x_i) = 0$. Hence, by induction on t, $\eta_t(x_i) = 0$ for $i = 1, 2, \dots$ and in particular $T(x) = \infty$.

If $A_{\eta_0}^v(x) = 0$, then $\eta_0(x) = 0$ and $T(x) = 0$.

Finally, to discuss the case $1 \leq A_{\eta_0}^v(x) < \infty$, set

$$L(x, k) = \{y \in \mathbb{Z}^d: (y_1 - x_1) + \cdots + (y_d - x_d) = k\}$$

and for every random field α,

$$B_\alpha(x, k) = C_\alpha^v(x) \cap L(x, k).$$

Now $B_\alpha(x, k) \neq \varnothing$ if and only if $0 \leq k \leq A_\alpha^v(x) - 1$. Therefore, every site $y \in B_{\eta_t}(x, A_{\eta_t}^v(x) - 1)$ has all its neighbors of the form $y + e_r$, $r = 1, \dots, d$, occupied at time t and, hence, $\eta_{t+1}(y) = 1$, so that $A_{\eta_{t+1}}^v(x) \leq A_{\eta_t}^v(x) - 1$. On the other hand, if $A_{\eta_t}^v(x) \geq 2$, then there must be a site $z \in B_{\eta_t}(x, A_{\eta_t}^v(x) - 2)$ such that one of its neighbors of the form $z + e_r$, $r = 1, \dots, d$, belongs to $C_{\eta_t}^v(x)$ and, therefore, is vacant at time t. Hence $\eta_{t+1}(z) = 0$, so that

$A^v_{\eta_{t+1}}(x) \geq A^v_{\eta_t} - 1$. In conclusion, $A^v_{\eta_t}$ decreases by exactly one unit each unit of time, until it reaches the value 0 at the time $T(x)$. This concludes the proof. □

As a corollary we have the following proposition.

PROPOSITION 4.2. *For the oriented models*

(i) $\rho(p) = 1 - \theta^v(p) = 1 - \theta^o(1-p)$.
(ii) $p_c = \pi_c = p_c^v = 1 - p_c^o$.
(iii) *If* $p > \pi_c$, *then the following limit exists*:

$$\gamma(p) = \lim_{a \to \infty} -\frac{1}{a} \log P(T > a) > 0.$$

(iv) $\rho(p_c) = 0$.

PROOF. (i), (ii) and (iii) follow from the previous proposition and the fact that in all dimensions, by the methods of Aizenman and Barsky (1987) or of Menshikov (1986) and Menshikov, Molchanov and Sidorenko (1986), if $p > p_c^v$, then $P(A^v_{\beta_p}(0) > a)$ decays exponentially with a. The existence of the limit in (iii) follows from standard supermultiplicity arguments [see, e.g., Durrett (1984)]. (iv) follows from (i) and (ii) and the fact that in any dimension $\theta^v(p_c^v) = 0$, which can be proven by a straightforward adaptation of the methods of Bezuidenhout and Grimmett (1990). □

So in $d \geq 2$, $0 < p_c = \pi_c < 1$. The region $0 < p < p_c$ is also worth study and, in particular, one can ask what properties the measure μ^p has and how fast the system converges to this measure. All these questions can be translated, thanks to Proposition 4.1, into related questions for oriented site percolation. For instance, in $d = 2$ one knows from Durrett and Griffeath (1983) [see also Durrett (1984); the adaptation from bond to site percolation is trivial] that if $p < p_c^v$, then $P(a < A^v_{\beta_p}(0) < \infty)$ decays exponentially with a. It follows that the convergence to μ^p occurs exponentially fast in this regime. The same can also be proven in higher dimensions, using the methods in Bezuidenhout and Grimmett (1990). Further information, for instance on critical exponents in two dimensions, can be extracted from Durrett, Schonmann and Tanaka (1989a, 1989b); the concept of graphical duality used there was adapted to oriented site percolation by Wierman (1985).

5. Results obtained by comparison with oriented models. In this section we will use techniques from the two previous sections to show that the oriented models can be used to divide the models defined in the Introduction into two classes with qualitatively different behaviors and to study their properties. The criterion for this partition is the property of dominating one of the oriented models. Again, in this section the dimension is arbitrary and is omitted in the notation.

Proposition 5.1. *If a model does not dominate any oriented model, then for this model* $p_c = \pi_c = 1$.

Proof. Recall the definitions

$$Q_k = \{x \in \mathbb{Z}^d: |x_i| \le k, i = 1, \dots, d\},$$
$$T = \inf\{t \ge 0: \eta_t(0) = 1\}.$$

Suppose that in the configuration η_t the cube Q_3 is completely vacant. The neighborhood $\mathcal{N}_x$ of each site $x \in Q_3$ intercepts $(Q_3)^c$ in a subset of a set of the form $\{a_1 e_1, \dots, a_d e_d\}$, where $a_r \in \{-1, +1\}$ for $r = 1, \dots, d$. Since the model does not dominate any oriented model, we will certainly have Q_3 completely vacant also in the configuration η_{t+1}. By induction on t, η_t leaves Q_3 completely vacant at all $t \ge 0$, provided it does it at $t = 0$. So for all $p < 1$,

$$P_p(T = \infty) \ge P_p(\eta_0(x) = 0 \text{ for every } x \in Q_3) = q^{3d} > 0. \qquad \square$$

Theorem 3.1 and Proposition 5.1 can be used to locate the critical points p_c and π_c of all bootstrap percolation models on $\mathbb{Z}^d$. In case $l \le d$, the model dominates the modified basic model and, hence, $p_c = \pi_c = 0$. If $l > d$, then the model does not dominate any oriented model and, therefore, $p_c = \pi_c = 1$.

We turn now to the models that dominate some oriented model. First we give some definitions. As in Section 3, we define the (d-dimensional) dynamics restricted to a set $\Gamma \subset \mathbb{Z}^d$ by freezing the states of all sites outside of Γ and letting the states inside of Γ evolve with the rules of the model we are considering. We will say that $\Lambda \subset \Gamma$ is completely covered by the dynamics restricted to Γ by the configuration η, if starting from the configuration η^Γ [defined by (3.1)] and letting the system evolve according to the dynamics restricted to Γ, Λ becomes eventually completely occupied. As we explained in Section 3, we only have to wait $|\Gamma|$ units of time to know whether this happens or not. Set

$S(N, p) = P(Q_N$ is completely covered by the dynamics restricted to Q_{2N} by a random configuration chosen according to a product measure with density $p)$.

For fixed N, $S(N, p)$ is a nondecreasing function of p; this motivates the definition

$$\bar{p}_c = \inf\left\{p \in [0,1]: \limsup_{N \to \infty} S(N, p) = 1\right\}.$$

Our main result in this section is the following.

Theorem 5.1. *If a model dominates one of the oriented models, then for this model*:

(i) $\pi_c = \bar{p}_c < 1$.

(ii) *There is a positive constant* C, *which may depend on the dimension, so that for* $t \ge 1$,

$$P_{\pi_c}(T > t) \ge Ct^{-d+1}.$$

Observe that for the models that do not dominate any oriented model, $P_{\pi_c}(T > t) = P_1(T > t) = 0$ and hence (ii) is false for these models. On the other extreme, for models such that $\pi_c = 0$, as the modified basic model, $P_{\pi_c}(T > t) = 1$, which is much stronger than (ii). Result (ii) implies for the models for which it applies that if the critical exponent κ exists, then $\kappa \le d - 1$.

In the proof of Theorem 5.1 we will use a renormalization procedure similar to (but somewhat different from) the one used in Section 3. As before, we fix a space scale N and to each site $k \in \mathbb{Z}^d$ in the renormalized lattice we associate the cube

$$Q_{N,k} = \{x \in \mathbb{Z}^d \colon x - (2N+1)k \in Q_N\}.$$

Again we say that the renormalized site k is occupied if all the sites of $Q_{N,k}$ are occupied in the original lattice. The renormalized time scale τ now will be defined by

$$t = |Q_{2N}| + (2dN+1)\tau = (4N+1)^d + (2dN+1)\tau.$$

The cardinality of Q_{2N} appears in this definition because we want to give a chance for each cube $Q_{N,k}$ to become completely occupied by the dynamics restricted to the cube $\overline{Q}_{N,k}$ of side $4N+1$, concentric with $Q_{N,k}$,

$$\overline{Q}_{N,k} = \{x \in \mathbb{Z}^d \colon x - (2N+1)k \in Q_{2N}\}.$$

Set

$$(5.1) \quad \beta(k) = \begin{cases} 1, & \text{if } Q_{N,k} \text{ becomes completely occupied by the dynamics} \\ & \text{restricted to } \overline{Q}_{N,k} \text{ by } \eta_0, \\ 0, & \text{otherwise.} \end{cases}$$

The random variables $\beta(k)$ are identically distributed, with

$$(5.2) \qquad P_p(\beta(k) = 1) = S(N, p),$$

but they are not independent. Anyhow, they have only a finite range of dependency structure, which is sufficient for our purposes. Indeed, if $k_1, k_2, \dots, k_n$ are such that $\|k_i - k_j\| > 2d$ (where $\|\cdot\|$ is the l_1 norm), for every $1 \le i < j \le n$, then the random variables $\beta(k_1), \dots, \beta(k_n)$ are mutually independent, since they depend on what happens in disjoint regions of the space and η_0 is a product random field.

By hypothesis we will consider a model that dominates some oriented model and with no loss of generality we can suppose that the dominated model is the basic oriented model. It turns out then that if the renormalized site k has at the renormalized time τ all its neighbors of the form $k + e_i$, $i = 1, \dots, d$, occupied, then at the renormalized time $\tau + 1$ it will be occupied (the renormalized dynamics also dominate the basic oriented model). A formal proof is provided now.

LEMMA 5.1. *For a model that dominates the basic oriented model, if at* $t = 0$ *the regions* $Q_{N,k}$, $k \in \{e_i, i = 1, \dots, d\}$ *are completely occupied, then at* $t = 2dN + 1$, $Q_{N,0}$ *will also be completely occupied.*

PROOF. Define the regions

$$S_l = \{x \in Q_{N,0} \colon x_1 + \cdots + x_d = l\}.$$

These regions are nonempty for $l = -dN, \dots, dN$ and $Q_{N,0}$ is equal to their union. Now observe that if $x \in S_l$, then for $i = 1, \dots, d$,

$$x + e_i \in S_{l+1} \bigcup \left(\bigcup_{k=1}^{d} Q_{N,k} \right).$$

Therefore at time $t \geq 1$, all the sets S_l, $l = dN, dN - 1, \dots, dN + 1 - t$, will be completely occupied and, in particular, at time $2dN + 1$, $Q_{N,0}$ will be completely occupied. □

PROOF OF THEOREM 5.1. As observed above, we suppose that the model dominates the basic oriented model. From the results in Section 4 it is clear that $\pi_c < 1$, so to prove (i) we have to show that

$$\pi_c = \bar{p}_c. \tag{5.3}$$

Given $p > \pi_c$, there are $\gamma > 0$ and $C < \infty$ such that

$$P_p(T > t) \leq Ce^{-\gamma t}. \tag{5.4}$$

But the argument used to prove (3.5), based on the fact that the effects travel with a maximum speed 1, can be used also here, giving

$$\begin{aligned} 1 - S(N,p) &\leq P_p\left(\bigcup_{x \in Q_N} \{\eta_{2N-N}(x) = 0\} \right) \\ &\leq |Q_N| \cdot P_p(T > N). \end{aligned} \tag{5.5}$$

From (5.4) and (5.5) we obtain

$$\lim_{N \to \infty} S(N,p) = 1.$$

Therefore, $p > \pi_c$ implies $p \geq \bar{p}_c$; that is,

$$\bar{p}_c \leq \pi_c. \tag{5.6}$$

To prove the complementary inequality we will use the renormalization procedure. Given $p > \bar{p}_c$ and $\varepsilon \in (0, d^{-2d})$, there is $N < \infty$ such that

$$S(N,p) > 1 - \varepsilon. \tag{5.7}$$

Consider the renormalized process constructed with this N. By attractiveness and the remarks made before the beginning of this proof, we know that at the renormalized time τ the state of occupancy of the renormalized sites dominates the state at time τ of the basic oriented model started at $\tau = 0$

from the random field ($\beta(k)$: $k \in \mathbb{Z}^d$), defined by (5.1). So, if $m = \lfloor (t - (4N+1)^2)/(2dN+1) \rfloor$, then from Proposition 4.1,

$$P_p(T > t) \le P_p\big(A^v_\beta(0) > m\big). \tag{5.8}$$

But if $A^v_\beta(0) > m$, then there is an oriented path $(x_1, \dots, x_m)$ with $x_1 = 0$ and $\beta(x_i) = 0$, $i = 1, \dots, m$. From the remarks made after the definition of β, the random variables $\beta(x_1)$, $\beta(x_{1+2d})$, $\beta(x_{1+4d}), \dots, \beta(x_{1+\lfloor m/2d \rfloor 2d})$ are mutually independent, and since there are d^{m-1} oriented paths as above, we have, using (5.2) and (5.7),

$$\begin{aligned} P_p\big(A^v_\beta(0) > m\big) &\le \big(P_p(\beta(0) = 0)\big)^{\lfloor m/2d \rfloor} d^{m-1} \\ &\le \varepsilon^{(m/2d)-1} d^{m-1} = C(\varepsilon, N)\big(\varepsilon^{1/2d} d\big)^{t/(2dN+1)}. \end{aligned} \tag{5.9}$$

Since $\varepsilon < d^{-2d}$, (5.8) and (5.9) imply that if $p > \bar{p}_c$, then $p \ge \pi_c$; that is,

$$\pi_c \le \bar{p}_c. \tag{5.10}$$

(i) follows from (5.6) and (5.10).

To prove (ii) we suppose, using (i), that $\bar{p}_c = \pi_c > 0$, since otherwise there is nothing to be proved. Set $\varepsilon_0 = d^{-2d}/2$. If there is N such that $S(N, \bar{p}_c) > 1 - \varepsilon_0$, then by continuity there is a $p < \bar{p}_c$ such that $S(N, p) > 1 - \varepsilon_0$ [observe that for each fixed N, $S(N, p)$ is a polynomial function of p]. But then, from the arguments above used to prove (5.10) we would have $\pi_c \le p < \bar{p}_c$, in contradiction with (5.6). So, for every N we have

$$S(N, \pi_c) \le 1 - \varepsilon_0. \tag{5.11}$$

Equations (5.5) and (5.11) are enough to prove a weaker form of (ii), with t^{-d} replacing t^{-d+1}. To prove the statement we made we need to strengthen (5.5). Set

$$D_N = \big\{x \in Q_N \colon x_i = N \text{ for some } i \in \{1, \dots, d\}\big\}.$$

D_N is the union of d of the $2d$ faces of Q_N. If D_N is completely occupied at time t, then Q_N will be completely occupied at time $t + (2dN + 1)$, since we are assuming that the model dominates the basic oriented model. So Q_N in (5.5) may be replaced by D_N and we obtain

$$1 - S(N, p) \le d(2N+1)^{d-1} P_p(T > N). \tag{5.12}$$

From (5.11) and (5.12), for every integer $N \ge 1$,

$$P_{\pi_c}(T > N) \ge \frac{\varepsilon_0}{d(2N+1)^{d-1}},$$

which implies (ii) and completes the proof. □

We end this section with the following remark, which provides an alternative way to describe the two classes into which the models have been divided. From the proof of Proposition 5.1 it is clear that for a model that does not dominate any oriented model, finite clusters of vacant sites can persist forever.

On the other hand, if a model dominates one of the oriented models (let us say the basic oriented model), then each finite cluster of vacant sites will eventually become completely occupied, since the sites of the cluster that maximize the sum of the coordinates at time t will become occupied at time $t+1$. In this case only infinite clusters can survive.

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