Random walks in random environments

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A renormalization-group analysis is carried out of the long-time behavior of random walks in an environment with a positionally random local drift force. It is argued that, independent of the strength of the disorder, the mean-square displacement, $\langle x^2(t)\rangle$, is linear in time (i.e., diffusive) for dimensions $d \gtrsim 2$. In two dimensions, universal $t/\ln t$ corrections are found and for $d=2-\epsilon$, the behavior is subdiffusive with $\langle x^2(t)\rangle \sim t^{1-\epsilon^2}$.

Marinari *et al.*[1] have recently argued that random walks in one-dimensional environments with local drifts which vary randomly as a function of the position of the walkers can give rise to $1/f$ noise. They argue that this arises from the logarithmic time dependence of the mean-square displacements of such walks found by Sinai.[2] They then go on to quote numerical evidence for a similar logarithmic time dependence of the displacement of random walks in similar but strongly disordered *two*-dimensional environments.[1]

In this paper we analyze the behavior as a function of dimension of random walks in environments with random local drifts with short-range correlations. In particular, a renormalization group expansion about the limit of weak disorder is carried out. We will argue that, contrary to the logarithmic time dependence suggested in Ref. 1, in any dimension greater than or equal to two, the mean-square displacement of random walks in such random environments will be *linear* in time at long times (i.e., normal diffusive behavior) *independent* of the strength of the disorder. In two dimensions, which is the upper critical dimension of the problem, there are universal logarithmic *corrections* to the diffusive long-time behavior.

Following Ref. 1, we consider a continuum random walk in d dimensions

$$\frac{d\vec{x}}{dt}=\vec{\eta}(t)+\vec{F}(\vec{x}) \tag{1}$$

with Gaussian-distributed noise η and random drift force F characterizing the environment with correlations

$$\begin{aligned}&\langle \eta^{\alpha}(t)\eta^{\beta}(t')\rangle=2D\delta^{\alpha\beta}\delta(t-t')\ ,\\ &\langle F^{\alpha}(\vec{x})F^{\beta}(\vec{x}\,')\rangle=\Delta\delta^{\alpha\beta}\delta(\vec{x}-\vec{x}\,')\ ,\\ &\langle \eta\rangle=\langle F\rangle=0\ .\end{aligned} \tag{2}$$

It is necessary to impose a short distance cutoff to make the problem well defined. The associated Fokker-Planck equation for the probability distribution $P(\vec{x},t)$ is

$$\frac{\partial P}{\partial t}=D\nabla^2 P-\vec{\nabla}\cdot(\vec{F}P)\ . \tag{3}$$

In order to calculate properties of the probability distribution averaged over the random force $\vec{F}$, we write a Martin-Siggia-Rose generating function[3,4]

$$\begin{aligned}Z_F\{h(\vec{x},t)\}&\equiv\prod_{t,\vec{x}}\int dP(\vec{x},t)\\ &\quad\times\delta\left[\frac{\partial P(\vec{x},t)}{\partial t}-D\nabla^2P+\vec{\nabla}\cdot(\vec{F}P)\right]\\ &\quad\times\exp\int dt\int d\vec{x}\,h(\vec{x},t)P(\vec{x},t)\\ &\equiv\prod_{t,\vec{x}}\int dP(\vec{x},t)\int\frac{d\hat{P}(\vec{x},t)}{2\pi}\exp\mathscr{L}_F(P,\hat{P})\\ &\quad\times\exp\int dt\int d\vec{x}\,h(\vec{x},t)\\ &\quad\times P(\vec{x},t)\ ,\end{aligned} \tag{4}$$

where we have introduced a conjugate field $\hat{P}(\vec{x},t)$ which acts like a source for random walks and the Lagrangian is (after an integration by parts)

$$\mathscr{L}_F=i\int dt\int d\vec{x}\left[\hat{P}(\vec{x},t)\left[\frac{\partial P(\vec{x},t)}{\partial t}-D\nabla^2 P\right]-P\vec{F}(\vec{x})\cdot\vec{\nabla}\hat{P}\right]\ . \tag{5}$$

By causality, $Z_F\{h=0\}=1$ for each configuration of the F's.[5] Therefore, correlation functions of the P's, which can be obtained by differentiating $Z_F\{h(\vec{x},t)\}$ with respect to the h's at $h=0$, do *not* need normalizing denominators.[4] Hence the averaging over F can be done immediately and averaged correlation functions obtained from derivatives of

$$\begin{aligned}Z\{h\}&=\prod_{\vec{x}}\int d\vec{F}(\vec{x})C\exp\left[\frac{-1}{2\Delta}\int d\vec{x}\,|\vec{F}(\vec{x})|^2\right]Z_F\{h\}\\ &\equiv\prod_{\vec{x},t}\int dP\int\frac{d\hat{P}}{2\pi}\exp(\mathscr{L})\end{aligned} \tag{6}$$

with C a normalizing constant, and

$$\mathscr{L}=\mathscr{L}_0-\frac{\Delta}{2}\int dt\int dt'\int d\vec{x}[P(\vec{x},t)\vec{\nabla}\hat{P}(\vec{x},t)\times P(\vec{x},t')\vec{\nabla}\hat{P}(\vec{x},t')]\ ,$$

where $\mathscr{L}_0$ is the bare Lagrangian,

$$\mathscr{L}_0 = i\int dt \int d\vec{x}\, \hat{P}(\vec{x},t)\left[\frac{\partial P(\vec{x},t)}{\partial t} - D\nabla^2 P(\vec{x},t)\right]$$

$$= i\int_\omega \int_{\vec{q}} \hat{R}(-\vec{q},-\omega)R(+\vec{q},\omega)(-i\omega+Dq^2)\ , \qquad (7)$$

where we have introduced the Fourier transforms $\hat{R},R$ of $\hat{P},P$.

In the absence of disorder, i.e., $\Delta=0$, we just have a conventional random walk with diffusive behavior at long times. The simplest question to address, initially, is the stability of this zero disorder diffusive fixed point to a small amount of disorder. In this formulation, power counting about this $\Delta=0$ diffusive fixed point is straightforward. We are interested in the response of the probability distribution at time t to a source of a random walk at earlier time t'. This is just the response function

$$-i\langle \hat{P}(\vec{x}\,',t')P(\vec{x},t)\rangle$$

which will vanish by causality for $t\le t'$. The Fourier transform response function

$$-i\langle \hat{R}(\vec{q}\,',\omega')R(\vec{q},\omega)\rangle = \delta(\vec{q}+\vec{q}\,')\delta(\omega+\omega')G(\vec{q},\omega) \qquad (8)$$

is trivially calculable for the $\Delta=0$ case,

$$G_0(\vec{q},\omega) = \frac{1}{-i\omega+Dq^2}\ . \qquad (9)$$

To calculate the relevancy or irrelevancy of Δ about the $\Delta=0$ fixed point, we rescale lengths by a factor b and take $R\to R'\zeta$, $\hat{R}\to\hat{R}'\hat{\zeta}$, and $\omega\to\omega' b^{-z}$. It is convenient to keep the form of the propagator (Eq. 9) fixed with $D'=D$, we thus have at the $\Delta=0$ fixed point, the dynamic exponent given by

$$z=2 \qquad (10)$$

and

$$\hat{\zeta}\zeta = b^{d+4}\ . \qquad (11)$$

The disorder Δ rescales as

$$\Delta' = b^{2-d}\Delta\ , \qquad (12)$$

and hence *weak* disorder is irrelevant for dimensions greater than two. To investigate the behavior near two dimensions, it is natural to try to do a renormalization group expansion in powers of Δ for weak disorder. It is straightforward to check that no other relevant or marginal operators are consistent with the symmetry.

The renormalization group expansion can be readily done diagramatically: the propagator iG_0 is represented by the line shown in Fig. 1(a) with the wiggly end the $\hat{R}$. The vertex which carries a factor $-\Delta/2$ is represented by Fig. 1(b) where the slashes denote $i\vec{q}$ on the incoming momentum and the dotted line carries zero frequency (but any momentum). Causality requires that all closed loops of G_0 propagators vanish.[5]

The lowest-order renormalization of the propagator

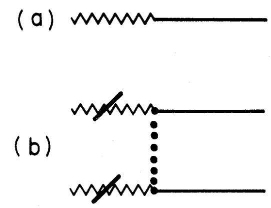

FIG. 1. (a) Line representing the propagator $iG_0(\vec{q},\omega)$. The wiggly line is the $\hat{R}$ end which has frequency $-\omega$. (b) Graph representing the vertex. The slashes denote $i\vec{q}$ on the incoming momentum and the dotted line carries zero frequency but any momentum.

would arise from the graphs shown in Figs. 2(a) and 2(b). However, the first vanishes by causality and the second under inversion of the internal momentum. The only second-order diagram which does not similarly vanish is shown in Fig. 2(c). It has a q^2 dependence on the external momentum at small q and hence can renormalize the $q^2\hat{R}R$ term in the Lagrangian; however, there is no renormalization of the coefficient of the $\omega\hat{R}R$ term. There are three diagrams which renormalize Δ to order Δ^2 shown in Fig. 3. The first [Fig. 3(a)] gives a contribution which *increases* Δ and the other two give a larger contribution which *decreases* Δ. Evaluation of the diagrams and rescaling the fields to keep $D'=D=1$ readily yields differential recursion relations,

$$\frac{d\Delta}{dl} = (2-d)\Delta - c\Delta^2 + O(\Delta^3)\ , \qquad (13)$$

$$z = 2 + f\Delta^2 + O(\Delta^3)\ , \qquad (14)$$

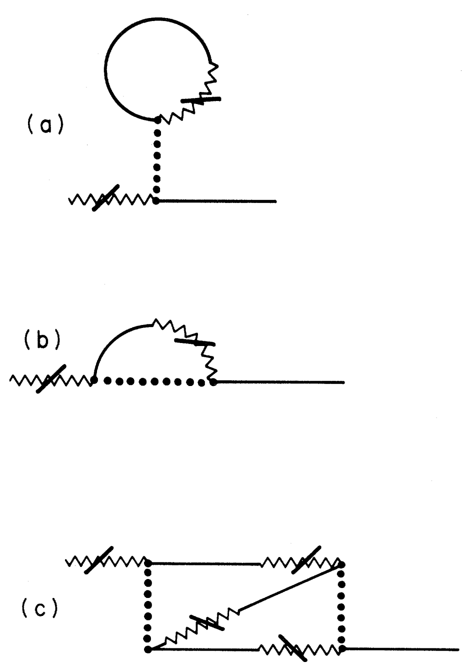

FIG. 2. Diagrams which renormalize the propagator; first-order diagrams (a) and (b) vanish; (c) is the only nonvanishing second-order term.

where the coefficients are given by

$$c=-\frac{1}{2\pi}+\frac{1}{4\pi}+\frac{1}{2\pi}=\frac{1}{4\pi} \tag{15}$$

[contributions from Figs. 3(a), 3(b), and 3(c), respectively]; and

$$f=\frac{1}{8\pi^2} .$$

In more than two dimensions, to second order in Δ the effective disorder thus becomes weaker at long-length scales. In less than two dimensions, small disorder will grow with length scale and there is a nontrivial *stable* fixed point of Eq. (13) in $d=2-\epsilon$ at

$$\Delta^*=\frac{\epsilon}{c}+O(\epsilon^2) . \tag{16}$$

At this fixed point the dynamic exponent is given from Eq. (14),

$$z=2+\frac{f}{c^2}\epsilon^2+O(\epsilon^3)$$
$$=2+2\epsilon^2 . \tag{17}$$

The field rescalings will only enter in the combination $\hat{\zeta}\zeta$; at a scale e^l, $\hat{\zeta}(l)\zeta(l)=e^{(d+2z)l}$. The impurity averaged mean-square displacement of a random walk started at $x=0$, $t=0$ is simply related to the response function by

$$\langle x^2(t)\rangle=-\frac{d}{dq^\alpha}\frac{d}{dq^\alpha}\int_\omega e^{-i\omega t}G(\vec{q},\omega)\Big|_{\vec{q}=\vec{0}} . \tag{18}$$

By scaling it follows that in $d=2-\epsilon$, the mean-square displacement of the random walk at long times will be given by

$$\langle x^2(t)\rangle\sim t^{2/z}$$
$$\sim t^{1-\epsilon^2} . \tag{19}$$

Note that the rescaling factor $\hat{\zeta}\zeta$ exactly cancels frequency and wave vector integrals in calculating $\langle x^2(t)\rangle$ (this is due to the linearity of the Fokker-Planck equation).

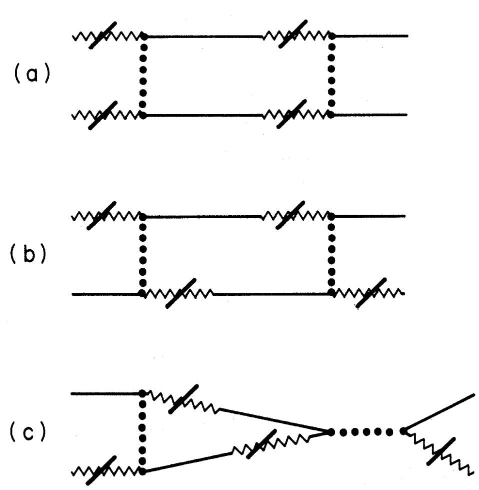

FIG. 3. Diagrams which renormalize the vertex.

We thus find that, at least for weak disorder, the long-time behavior of the displacement in less than two dimensions is *slower* than diffusive, as might be expected physically. In one dimension, Sinai[2] has proved that $\langle x^2(t)\rangle\sim(\ln t)^4$ at long times *independent* of the strength of the disorder. Presumably the exponent $2/z$ determining the long-time behavior in Eq. (19) goes smoothly from its value $1-\epsilon^2$ near two dimensions to zero, i.e., logarithmic behavior, in $d=1$. Since in exactly two dimensions the disorder is marginal, we might expect more complicated behavior there.

In $d=2$, we have

$$\frac{d\Delta}{dl}=-c\Delta^2 , \tag{20}$$

whence in terms of the bare Δ_0,

$$\Delta(l)=\frac{\Delta_0}{1+c\Delta_0 l} . \tag{21}$$

The characteristic frequency scale Ω will renormalize as

$$\frac{d\Omega}{dl}=z\Omega$$
$$=[2+f\Delta^2(l)]\Omega(l) \tag{22}$$

and we conclude that

$$\ln\left[\frac{\Omega(l)}{\Omega_0}\right]=2l-\frac{f}{c}\left[\frac{\Delta_0}{1+\Delta_0 cl}-\Delta_0\right] . \tag{23}$$

At sufficiently long times t, with corresponding bare frequency $\Omega_0=1/t$, the mean-square displacement, will, by scaling, be the square of the length scale e^l at which $\Omega(l)$ becomes of order one. From Eq. (23) we can solve asymptotically for l with $\Omega(l)=1$, Ω_0 very small and Δ_0 fixed and small. We conclude that, at sufficiently long times,

$$\langle x^2(t)\rangle\simeq 4D_R t\left[1+\frac{2f}{c^2\ln t}\right]=4D_R t\left[1+\frac{4}{\ln t}\right] , \tag{24}$$

where D_R is the renormalized diffusion constant and the *correction coefficient* $2f/c^2$ is *universal.* Thus in two dimensions there is diffusion at long times with universal slow transient $t/\ln t$ corrections.

Note that the diffusive behavior in two dimensions is due to the *absence* of frequency renormalization at order Δ. If an order Δ term *had* been present in Eq. (14), the mean-square displacement in $d=2$ would have behaved subdiffusively as $t/(\ln t)^\alpha$ (with some α) at long times. This type of behavior is found in related two dimensional models.[6]

In more than two dimensions, the irrelevancy (to second order in Δ) of the disorder about the pure diffusive fixed point implies that at least for weak disorder there will be diffusive behavior, i.e., $\langle x^2(t)\rangle\sim t$. A word of caution is called for in interpreting the result in $d\geq 2$. While the scaling at the diffusive fixed point does imply by itself that $\langle x^2(t)\rangle\sim t$ at long times, it does *not* immediately imply that the averaged probability distribution $\langle P(x,t)\rangle$ of a walk started at (0,0) behaves diffusively (i.e., like a

Gaussian) at long times for *all* finite, but arbitrarily large, x^2/t.

So far, our results are only valid for weak disorder. From the behavior in this limit, and the local renormalization group flows of Δ, it is tempting to immediately conclude that the stable zero disorder fixed point controls the long-time behavior for *all* Δ in $d \geq 2$ (i.e., the behavior is always diffusive) and that the stable weak disorder fixed point Δ^* controls the long-time behavior for all Δ in $d \lesssim 2$. This is consistent with the behavior in one dimension found by Sinai;[2] the $\langle x^2(t) \rangle \sim \ln^4 t$ result applies for his model independent of the strength of the disorder, as long as it is finite.

Support for this argument that the domain of attraction of the weak disorder fixed point includes all finite strength of the disorder is provided by the naive flow for strong disorder. If we consider putting the walk on a lattice with nearest-neighbor steps with a nonsingularly distributed probability bias representing $\vec{F}$, then the limit of infinite disorder corresponds to deterministic motion with the direction of flow out of each site random in space but fixed in time. In one dimension, it is clear that after t steps, in all but an exponentially small fraction of the ensemble of environments the walk will be trapped hopping between two neighboring sites. Thus $\langle x^2(t) \rangle$ will go to a *constant* at long times. In higher dimensions, since the probability per unit time of getting trapped in a small cycle (although there will also be trapping in large cycles) will be roughly independent of time, it is reasonable to assume that the configuration averaged mean-square displacement will also go to a constant at long times.

In one dimension, any amount of thermal noise (i.e., the disorder large but finite) will cause the mean-square displacement to diverge at long times;[2] the infinite disorder fixed point thus must be unstable to $1/\Delta$. In higher dimensions, the large trapped cycles and their domains of attraction will be convoluted and it is possible that at some point on a cycle with a large basin of attraction a small amount of thermal noise could take the walk onto another cycle, in contrast to the case in $d=1$ where a large basin of attraction of a cycle immediately yields (since in $d=1$, F can be thought of as the gradient of a potential) a large barrier which must be overcome by the thermal noise. Thus, naively, a small amount of thermal noise is unlikely to have a smaller effect on the long-time behavior in higher dimensions than in one dimension and hence the infinite disorder fixed point will generally be unstable to $1/\Delta$ in all dimensions.

Unless there are two (or more) transitions as a function of Δ, it is reasonable to believe that the flow away from $\Delta=\infty$ will go towards the stable weak disorder (in $d \lesssim 2$) or zero disorder (in $d \geq 2$) fixed point. If this is the case the weak disorder behavior investigated here will be valid for arbitrary finite strength of the disorder. Liang *et al.*[7] have recently pointed out, however, that if the distribution of hopping probabilities $p(i \rightarrow j)$, in a lattice model (with i,j nearest neighbors) is sufficiently singular, then subdiffusive behavior is possible in any dimension. This will occur when there are a large number of sites for which the probability of hopping out in *one* of the directions is very close to unity, i.e., if the distribution of the $p(i \rightarrow j)$ is strongly divergent as $p(i \rightarrow j)$ approaches one. In this case there will be neighboring i,j sites with $p(j \rightarrow j)$ and $p(j \rightarrow i)$ both near one and walks arriving at i or j will stay trapped for a long time, leading[7] to subdiffusive behavior. However as long as the distribution of the $p(i \rightarrow j)$ is nonsingular, which should be the case for discretizations of continuum models such as that discussed here, then this local trapping should not occur often enough to prevent most of the walks from diffusing.

In more than two dimensions, another suggestive argument can be made for diffusive behavior independent of the strength of the nonsingular disorder by drawing analogies with results on phonon localization. We consider the Fokker-Planck equation (3), which has the form

$$\frac{\partial P}{\partial t} = -JP \tag{25}$$

with J the non-Hermitian operator,

$$J = -D\nabla^2 + \vec{\nabla}\cdot\vec{F} \ . \tag{26}$$

The Green's functions of J are simply related to those of the adjoint operator,

$$J^+ = -D\nabla^2 - \vec{F}\cdot\vec{\nabla} \ , \tag{27}$$

which we note trivially has an extended eigenfunction (the constant function) with eigenvalue zero. A somewhat similar *Hermitian* operator,

$$\mathscr{H} = -D\nabla^2 - V(\vec{x})\nabla^2 - [\vec{\nabla} V(\vec{x})]\cdot\vec{\nabla} \ , \tag{28}$$

with $V(\vec{x})$ random (but bounded below by $-D$) has been studied in connection with phonon localization;[8] it also has a trivial extended eigenfunction with zero eigenvalue in any dimension. Renormalization group arguments near two dimensions have shown that for $d>2$ there is a *finite* range of eigenvalues E near zero for which the eigenfunctions of $\mathscr{H}$ are extended;[8] the disorder is effectively smaller at low energy, E. This is a consequence of the apparently reasonably general result that, in more than two dimensions, extended eigenfunctions of Hermitian operators which have the form $\mathscr{H} = -D\nabla^2$ plus short-range random parts stay extended as any parameter controlling the effective disorder is varied by a small amount.[9]

It is tempting to hope that this applies also to non-Hermitian operators which exhibit an extended state. If this naive assumption is correct, then J^+ should have extended states for a band of eigenvalues near zero. If J^+ were Hermitian, any initial condition could be expanded in eigenfunctions and the long-time behavior would be given by the extended eigenfunctions corresponding to the eigenvalues with the lowest real part; the long-time behavior would then be diffusive. This argument, though suggestive, *cannot* be made for the non-Hermitian case. The operator J^+ will generally *not* have a complete set of eigenfunctions even though it can straightforwardly be shown that the real parts of the eigenvalues of J^+ must be non-negative. It is plausible, however, that an argument for diffusive behavior of the eigenfunctions of J^+ (or J) can be made and a convincing demonstration of diffusive behavior for *strong* disorder in $d>2$ given by means similar to those of Ref. 8. This possibility certainly merits further study.

We note, however, that while the existence of extended states at low frequencies is likely to be *sufficient* to yield diffusive behavior of the one-particle Green's function of interest here, it is definitely *not* necessary. In particular, if the force $\vec{F}$ in Eq. (1) is the gradient of a random potential with mean square Δ_v, then calculations similar to those presented here show that Δ_v is irrelevant, implying diffusive behavior, even in one dimension where the disorder is surely sufficient to localize almost all the eigenfunctions. What is important for determining whether there is diffusive behavior, is not whether the eigenfunctions are localized, but how rapidly the localization length grows as the eigenvalue tends to zero. The arguments given here all support the conclusion that independent of the strength of the disorder random walks in nonsingular random environments will exhibit diffusive behavior in more than two dimensions.

The conclusion that there will always be diffusive behavior also *in* two dimensions is, however, inconsistent with the numerical results quoted in Ref. 1 for strong disorder. A possible source of this discrepancy is the slow transient in $d=2$, which causes the universal logarithmic correction to the diffusive behavior, and, on intermediate time scales, can yield an apparent exponent $z>2$ which varies very slowly with time. Tentative support for this interpretation is provided by recent numerical work.[7] However, another possible resolution of the apparent discrepancies between the continuum results here and the numerical lattice results of Ref. 1 for very strong disorder has been suggested by Liang *et al.*[7] In the strong disorder limit, the lattice model of Ref. 1 has a singular distribution of hopping probabilities $p(i\rightarrow j)$ of the form discussed above. This could cause the behavior of $\langle x^2(t)\rangle$ to be a subdiffusive power law of t.[7] As mentioned previously this is not expected to occur in the continuum with Gaussian (or other well-behaved) distribution of random forces nor in lattice models without anomalous distributions of the local hopping probabilities. A transition to subdiffusive behavior for strong disorder in $d\geq 2$ cannot, however, be ruled out entirely even for lattice models *without* anomalous distributions of local hopping probabilities.

Unfortunately, the hope of the authors of Ref. 1 that random walks in random environments might provide a *general* source of $1/f$ noise appears unlikely to be valid.

After a preliminary version of this manuscript had been written, the author received several preprints[10–12] which contained considerable overlap with this work. One of these[10] considers a more general problem but concludes that for the case of interest here there is diffusive behavior in $d\geq 2$, although neither the logarithmic corrections in $d=2$ nor the $O(\epsilon^2)$ correction in $2-\epsilon$ are calculated. A second preprint,[11] takes a very similar approach to that taken here, and the results in a new version agree with those presented here. A third paper by J. M. Luck,[12] evaluates the behavior in $d=2-\epsilon$, and finds logarithmic corrections to another quantity (related to the response to a uniform applied drift) in $d=2$. The result for the $O(\epsilon^2)$ correction to z in $2-\epsilon$ was smaller by a factor of 2 in the preprint than that derived here and larger by a factor of 2 in the published version. It is not clear what is the source of this discrepancy. Luck's analysis of the non-linear response to a uniform applied drift force generalizes to dimensions near two the one-dimensional results in Ref. 2.

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