Anomalous diffusion in steady fluid flow through a porous medium

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Diffusion in steady fluid flow through a porous medium with large fluctuations in the pore diameters is studied. For incompressible fluids, there are singular corrections to hydrodynamics in three dimensions and logarithmic divergences in $d=2$. When the constraint of incompressibility is relaxed, we recover recent results for diffusion in a random environment.

I. INTRODUCTION

It has long been known that convective coupling of a scalar contaminant to a sufficiently chaotic velocity field can dramatically enhance diffusion. Strong turbulence, for example, can decrease mixing times by many orders of magnitude. Even in equilibrium, thermal fluctuations in the velocity lead to long-time tail anomalies in the hydrodynamics of diffusion in dimensions $d>2$ and logarithmic divergences in $d=2$.[1] A more striking breakdown of hydrodynamics occurs for dimensions $d<2$.[2]

Turbulent and thermal fluctuations are chaotic in both space and time. One might ask just how chaotic a flow need be to significantly effect the long wavelength properties of diffusion. In this paper we investigate diffusion in the presence of a spatially chaotic but time-independent velocity field and show that spatial chaos by itself modifies the infrared properties of diffusion much as thermal fluctuations do.

The physical system which we have in mind is a porous medium through which fluid is pumped at a slow, constant rate. We shall examine how a passive contaminant moves under conditions of steady, laminar flow. A crude, microscopic model of such a system is a network of nodes connected by pipes of random length and diameter through which fluid is steadily flowing (Fig. 1). The random pipe diameters and directions lead to static velocity fluctuations about the mean-flow velocity. We shall be interested in diffusion on length scales that are large compared to the spacing between pipes. We shall further assume that fluctuations in the pore diameters are large, so that there are correspondingly large fluctuations in the local flow velocity about its mean value.

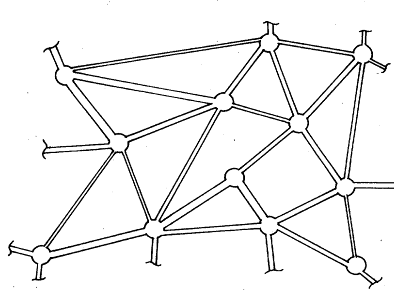

FIG. 1. Portion of a two-dimensional network of randomly placed nodes connected by pipes of random thickness.

To study the hydrodynamic limit, note first that in the absence of diffusion, the equation

$$\frac{\partial}{\partial t}\psi+\vec{\nabla}\cdot(\psi\vec{v})=0 \tag{1}$$

expresses the conservation of ψ, given that the convective ψ current is $\psi\vec{v}$. In the above, ψ is a coarse-grained concentration of material, and $\vec{v}$ is the microscopic velocity field averaged similarly. Adding diffusion to the problem leads to the modified equation

$$\frac{\partial}{\partial t}\psi+\vec{\nabla}\cdot(\psi\vec{v})=\vec{\nabla}\cdot(D\vec{\nabla}\psi)\ , \tag{2}$$

where the diffusion current has been expanded in gradients of ψ, and $D(\vec{x})$ is the diffusion constant which in principle may have the form $D(\vec{x})=D+\Delta(\vec{x})$, where $\Delta(x)$ is a random variable of mean zero. We will later argue that both Δ and the higher-order gradient terms are irrelevant to the long wavelength physics in the renormalization-group sense, as are all cumulants of $\vec{v}$ higher than the second. We shall work in the limit of small mean flow velocities, which allows us to neglect anisotropies in diffusion parallel and perpendicular to the flow. Accordingly, we investigate the model

$$\frac{\partial}{\partial t}\psi+\vec{v}_0\cdot\vec{\nabla}\psi+\vec{\nabla}\cdot(\psi\vec{v})=D\nabla^2\psi\ , \tag{3}$$

where $\vec{v}(\vec{x})$ is a Gaussian random variable with zero mean, and $\vec{v}_0$ is the average velocity. For $\vec{v}_0$ much less than the speed of sound, one usually expects the fluid to be virtually incompressible so that

$$\langle v_j(\vec{k})v_l(\vec{k}\,')\rangle=2f(k)P^T_{jl}(\vec{k})(2\pi)^d\delta(\vec{k}+\vec{k}\,')\ , \tag{4}$$

where P^T_{jl} is the transverse projection operator, $P^T_{jl}(\vec{k})=\delta_{jl}-k_jk_l/k^2$, and the δ function expresses the translation invariance of the velocity-velocity correlation function. It can be shown that if $f(k)=F+O(k^2)$, only F is a relevant variable at long wavelengths. Accordingly, we set $f(k)=F$. The effect of small deviations from incompressibility is discussed below.

Our model is related to the continuum limit of the "general nonsymmetric hopping model" introduced by Derrida and Luck.[3] Their model describes the diffusion of particles on a cubic lattice where the probability for hopping from site A to site B is not equal to that of going from B to A. After identifying a parameter $\vec{v}_0$ as the average rate of particle flow, Derrida and Luck show that the theory is singular in the $\vec{v}_0 \to 0$ limit in $d=2-\epsilon$ dimensions. Luck[4] then argues that the continuum model

$$\frac{\partial}{\partial t}P+\vec{v}_0\cdot\vec{\nabla}P+\vec{\nabla}\cdot(P\vec{v})=\nabla^2 P \ ,$$
$$\langle v_j(\vec{k})v_l(\vec{k}\,')\rangle=2F\delta_{jl}(2\pi)^d\delta(\vec{k}+\vec{k}\,') \quad (5)$$

reproduces the most singular part of the hopping model in perturbation theory as $\vec{v}_0 \to \vec{0}$. Thus (5) is the correct continuum limit of the hopping model. Model (5) has also been recently analyzed by Fisher[5] in the case of $\vec{v}_0=\vec{0}$. One immediately sees that (5) describes diffusion in a frozen random velocity field which is so compressible that the transverse and longitudinal fluctuations in $\vec{v}$ are given equal weights. We interpolate between our model and (5) by introducing a generalized model:

$$\left[\frac{\partial}{\partial t}+\vec{v}_0\cdot\vec{\nabla}-D\nabla^2\right]\psi=-\lambda^T\vec{\nabla}\cdot(\vec{v}^{\,T}\psi)-\lambda^L\vec{\nabla}\cdot(\vec{v}^{\,L}\psi) \ , \quad (6)$$

$$\langle v_l^T v_j^L\rangle=0 \ , \quad (7a)$$
$$\langle v_l^{T,L}(\vec{k})v_j^{T,L}(\vec{k}\,')\rangle=2F^{T,L}P_{lj}^{T,L}(\vec{k})(2\pi)^d\delta(\vec{k}+\vec{k}\,') \ ,$$

$$P_{jl}^T=\delta_{jl}-k_jk_l/k^2 \ , \quad P_{jl}^L=k_jk_l/k^2 \ , \quad (7b)$$

where λ^T and λ^L are dimensionless coupling constants introduced to keep track of the strengths of the couplings between the velocity fluctuations and ψ, while F^L and F^T measure the absolute size of the transverse and longitudinal fluctuations. This case of diffusion in a highly compressible fluid might be physically realized by pumping gas through vycor. By varying the gas density, the compressibility and hence the ratio of F^T to F^L can be changed.

Generally, the quantities of interest in this theory can be defined as follows. Let $\psi(\vec{k},\omega)$ be the Fourier transform of $\psi(\vec{x},t)\theta(t)$, where $\theta(t)$ is the unit step function, and let $\psi_0(\vec{k})=\psi(\vec{k},t=0)$. Then for $\vec{v}(\vec{k})\equiv\vec{0}$,

$$\psi(\vec{k},\omega)=\frac{\psi_0(\vec{k})}{-i\omega+i\vec{v}_0\cdot\vec{k}+Dk^2} \ . \quad (8a)$$

In the presence of velocity fluctuations, we define $D_R(\vec{k},\omega)$ and $\vec{v}_R(\vec{k},\omega)$ by

$$\langle\psi(k,\omega)\rangle\equiv\psi_0(k)/(-i\omega+i\vec{v}_R\cdot\vec{k}+k^2D_R) \ . \quad (8b)$$

Accordingly, $k^2D_R(\vec{k},\omega)+i\vec{k}\cdot\vec{v}_R$ is the Fourier transform of the diffusion kernel of the effective integral equation for $\langle\psi(\vec{k},t)\rangle$. The more physical quantities to calculate are the normalized moments of $\langle\psi(\vec{x},t)\rangle$. Let $\psi_0(\vec{k}=\vec{0})=\int d\vec{r}\,\psi_0(\vec{r})\equiv\overline{\psi}_0$ and define the normalized moments of $\langle\psi(\vec{x},t)\rangle$:

$$\bar{x}_\mu(t)=\int d\vec{x}\,x_\mu\langle\psi(\vec{x},t)\rangle/\overline{\psi}_0\equiv tv_R^\mu(t) \ , \quad (9a)$$

$$\bar{x}_{\mu\nu}(t)=\int d\vec{x}\,x_\mu x_\nu\langle\psi(\vec{x},t)\rangle/\overline{\psi}_0 \ , \quad (9b)$$

$$\Delta\bar{x}_{\mu\nu}(t)=\bar{x}_{\mu\nu}(t)-\bar{x}_\mu(t)\bar{x}_\nu(t)\equiv 2tD_R(t)\delta_{\mu\nu} \ . \quad (9c)$$

For the case $\vec{v}(\vec{k})=\vec{0}$, standard diffusion in a system moving at velocity $\vec{v}_0$ results, and we can easily show that

$$\vec{v}_R=\vec{v}_0 \ , \quad D_R=D \ . \quad (10)$$

With the randomness added, one expects the behavior of $D_R(t)$ and $\vec{v}_R^{\,\mu}(t)$ to be more complex.

As in the special case $\lambda^T=\lambda^L, F^T=F^L$ investigated by Luck[4] and Fisher,[5] we find in general that perturbation theory breaks down below two dimensions. To see this, consider an expansion of $D_R\equiv D_R(\vec{k}=\vec{0},\omega=0)$ for the case of $\vec{v}_0=\vec{0}$ to $O(\lambda^2)$. One can easily show that

$$D_R=D+2\left[\frac{d-1}{d}\right]\frac{\lambda_T^2F_T}{D}\int\frac{d^2q}{q^2}-\frac{2}{d}\frac{\lambda_L^2F_L}{D}\int\frac{d^2q}{q^2}+O(\lambda^4) \ . \quad (11)$$

Note that that transverse fluctuations promote mixing (i.e., D increases), while longitudinal fluctuations cause D to decrease. For $d\le 2$, the coefficients of λ_T^2 and λ_L^2 diverge and perturbation theory breaks down. Because the transverse and longitudinal terms come in with opposite signs, cancellations can occur, and (11) is indeterminant. One needs a renormalization group to sort out the true, asymptotic behavior.

We find that the general model (6) can exhibit behavior quite different from that reported by Luck[4] and Fisher[5] for model (5). Let $\bar{\lambda}_{T,L}=\lambda_{T,L}(F_{T,L}/D)^{1/2}$. Below two dimensions, the isotropic $\bar{\lambda}_L=\bar{\lambda}_T=\sqrt{2\pi\epsilon}$ fixed point analyzed by Fisher remains stable in the more general context of model (6), and will control the very long-time diffusive behavior $D_R(t)$ of any fluid which starts with both $\bar{\lambda}_L$ and $\bar{\lambda}_T$ nonzero. However, there is an additional fixed point at $\bar{\lambda}_T=\sqrt{\pi\epsilon}$, $\bar{\lambda}_L=0$ which describes an incompressible fluid. For $\bar{\lambda}_L \ll \bar{\lambda}_T$, this fixed point will control $D_R(t)$ for intermediate times $t \ll t_c \sim e^{\bar{\lambda}_T^2/\bar{\lambda}_L^4}$ in $d=2$. In $d=2$, for instance, this transverse fixed point leads to the "super-diffusive" behavior, $D(t)\sim D(\ln t)^{1/2}$ for an incompressible fluid, in contrast to the result $D(t)\sim D(1+4/\ln t)$ reported by Fisher[5] for isotropic velocity fluctuations.

Even without a renormalization-group analysis, one can see a lot of what is happening physically in the fluid directly from Eq. (11). Just as in turbulence, the transverse fluctuations shear the fluid, and enhance diffusion. In contrast, longitudinal fluctuations tend to suppress diffusion. This suppression can be explained by considering what happens when a compressible fluid runs into a particularly constricted part of the network of pipes. Because the fluid is compressible, its density will be driven up and a "traffic jam" develops. Because the contaminant gets stuck in the "jams" convective mixing becomes less efficient, and the diffusion constant is reduced. Equation (11) shows that the actual diffusion is the result of a competition between shear mixing and traffic jams.

The effect of nonzero $\vec{v}_0$ is to ultimately cut off the logarithmic singularities discussed above. To see this, note that because the random field has very short-range correlations, one would expect that any anomalous diffusive behavior is caused by a blob of material remaining in the same region long enough to interact strongly with the same quenched random velocity. If the blob is being quickly convected along by $\vec{v}_0$, it will sample a region which is large compared to the correlation length of the randomness in the time the blob takes to appreciably diffuse. In the limit of large $\vec{v}_0$, effects due to the random velocity field should cancel out. Equivalently, if we observe the system on a timescale long enough for the convection by v_0 to matter, we should see conventional diffusive behavior. If we look at shorter times, we should see anomalous diffusion caused by the randomness. As a rough guide, the convective flow $\vec{v}_0$ should become important at a time τ such that the rms distance a blob has diffused is equal to the distance it has convected:

$$\tau = D v_0^{-2} \ . \tag{12}$$

We indeed find this crossover behavior for asymptotically small v_0 and $1/t$. Criteria (12) is valid for $d \geq 2$. Below the critical dimension $d=2$ there are $O(\epsilon)$ corrections to the exponent -2.

II. RENORMALIZATION GROUP

To analyze model (7), we first Fourier transform using the conventions

$$f(\vec{k},\omega) = \int d\vec{x}\, dt\, e^{i(\omega t - \vec{k}\cdot\vec{x})} f(\vec{x},t) \ , \tag{13}$$

$$\psi(\vec{x},t) \equiv \theta(t)\psi(\vec{x},t) \ , \tag{14}$$

$$\int_q \equiv \int d\vec{q}\, \frac{1}{(2\pi)^d} \ , \quad |q| \leq \Lambda \ . \tag{15}$$

Here, the Λ is a cutoff initially of order a_0^{-1}, where a_0 is a microscopic length scale. Equation (7) now becomes

$$\psi(\vec{k},\omega) = \frac{1}{-i\omega + i\vec{k}\cdot\vec{v}_0 + Dk^2} \left[\psi_0(\vec{k}) + \frac{\vec{k}}{i} \cdot \int_q [\lambda^T \vec{v}^{\,T}(\vec{k}-\vec{q}) + \lambda^L \vec{v}^{\,L}(\vec{k}-\vec{q})] \psi(\vec{q}) \right] . \tag{16}$$

which can be written graphically as in Fig. 2(a). The formal iterative solution of (16) is shown in Fig. 2(b). The series obtained by averaging over the velocity field is displayed in Fig. 2(c). Let $\sum(k,\omega)$ be the sum of the one-particle irreducible graphs with one arrowed line running through them [Fig. 2(d)]. Then the standard resummation using geometric series gives

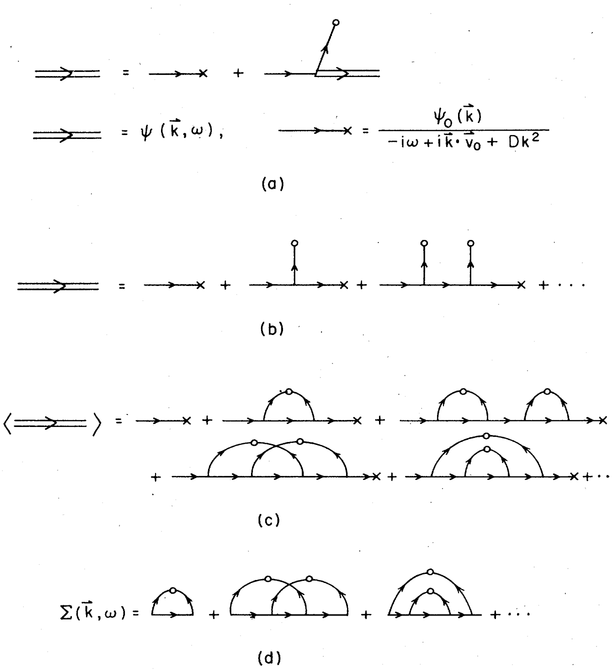

FIG. 2. (a) Graphical representation of Eq. (15). (b) Graphical solution of Eq. (15) by iteration. (c) Average over the velocity fluctuations in Fig. 2(b). (d) Graphs contributing the self-energy $\sum(\vec{k},\omega)$.

$$\langle \psi(\vec{k},\omega) \rangle = \frac{\psi_0(\vec{k})}{-i\omega + Dk^2 + i\vec{k}\cdot\vec{v}_0 - \sum(k,\omega)} \ . \tag{17}$$

Thus $\sum(k,\omega)$ directly yields the renormalized velocity and diffusion kernels. The upper critical dimension of the theory is suggested by evaluating the leading contribution to $\sum(k,\omega)$. For $d > 2$ naive perturbation theory works in the $\vec{k} \to \vec{0}$, $\omega \to 0$, $\vec{v}_0 \to \vec{0}$ limit, while for $d \leq 2$ the loop integrals are infrared divergent.

Following methods developed by Forster *et al.*,[2] we formulate a renormalization group to calculate properties of $\langle \psi \rangle$ for d near 2. First we set $\Lambda = 1$ for simplicity. We define a family of effective theories parametrized by l by averaging all components of $\vec{v}(\vec{k})$ such that $e^{-l} \leq |\vec{k}| \leq 1$ and then rescaling. We find that D and $\vec{v}_0$ are renormalized by the graph in Fig. 3(a), $\lambda^{L,T}$ by the graph in Fig. 3(b), and $F^{L,T}$ by the graphs in Fig. 3(c). We rescale using

$$k' = e^l k \ , \quad \omega' = \exp\left[\int_0^l z(l')dl'\right]\omega \ , \tag{18a}$$

$$\psi'(k',\omega') = \psi(k,\omega),$$

$$\vec{v}^{\,L,T}(\vec{k}) = \exp\left[\int_0^l w^{L,T}(l')dl'\right] \vec{v}^{\,\prime L,T}(\vec{k}^{\,\prime}) \ . \tag{18b}$$

Now, we set $\epsilon = 2 - d$, and introduce rescaled couplings

$$\bar{\lambda}_L = \frac{\lambda_L \sqrt{F_L}}{D} \ , \quad \bar{\lambda}_T = \frac{\lambda_T \sqrt{F_T}}{D} \ . \tag{19}$$

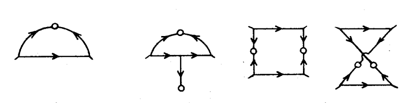

FIG. 3. (a) Graph which renormalizes $D(l)$ and $v_0(l)$ to $O(\epsilon)$. (b) Graph which renormalizes $\lambda_L(l)$ and $\lambda_T(l)$ to $O(\epsilon)$. (c) Graphs which renormalize $F^L(l)$ and $F^T(l)$ to $O(\epsilon)$.

To $O(\epsilon,\bar\lambda_T^2,\bar\lambda_L^2)$, no new interactions are generated. After choosing $z(l)$ and $w^{L,T}(l)$ to fix $D(l)$ and $F^{L,T}(l)$ at their initial values, we find the recursion relations

$$\frac{d\vec{v}_0}{dl}=(1-A_d\bar\lambda_T^2)\vec{v}_0 , \quad (20a)$$

$$\frac{d\bar\lambda_L}{dl}=\tfrac{1}{2}\epsilon\bar\lambda_L-\frac{A_d}{2}\bar\lambda_T^2\bar\lambda_L^2 , \quad (20b)$$

$$\frac{d\bar\lambda_T}{dl}=\tfrac{1}{2}\epsilon\bar\lambda_T-A_d\bar\lambda_T^3+\frac{A_d}{2}\bar\lambda_L\bar\lambda_T^2 , \quad (20c)$$

and

$$z(l)=2+B_d\bar\lambda_L^2-A_d\bar\lambda_T^2 , \quad w^{L,T}(l)=\frac{d}{2}+O(\bar\lambda^2) . \quad (21)$$

The constants A_d and B_d are related to the surface area of a d-dimensional sphere, $s_d=2\pi^{d/2}/\Gamma(d/2)$, by

$$A_d=2(d-1)S_d/d(2\pi)^d , \quad B_d=2S_d/d(2\pi)^d . \quad (22)$$

Note that $A_d=B_d=1/2\pi$ in two dimensions. A discussion of irrelevant variables is given in the Appendix.

The recursion relations (20) can be used to analyze the flows in the $(\bar\lambda_T)^2,(\bar\lambda_L)^2$ plane. Above $d=2$, there is a stable fixed point at $\bar\lambda_T=\bar\lambda_L=0$. Thus, the randomness is irrelevant and the theory is diffusive. The $d=2-\epsilon$ flow diagram is drawn in Fig. 4. There are two new fixed points, a stable "isotropic" fixed point at $\bar\lambda_L=\bar\lambda_T=\sqrt{2\pi\epsilon}$, and an unstable "transverse" fixed point at $\bar\lambda_L=0,\bar\lambda_T=\sqrt{\pi\epsilon}$. For large enough l, any initial condition with both $\bar\lambda_L$ and $\bar\lambda_T$ nonzero flows to the isotropic fixed point. If $\bar\lambda_L=0$, there the flow remains on the $\bar\lambda_T$ axis and flows

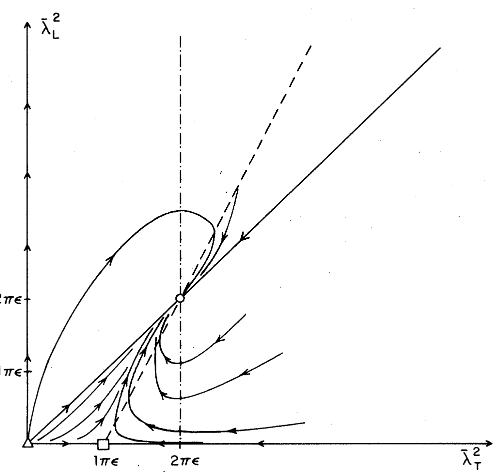

FIG. 4. Flow diagram in the $(\bar\lambda_L^2,\bar\lambda_T^2)$ plane for $d=2-\epsilon$. $\bar\lambda_L^2$ ($\bar\lambda_T^2$) is vertical along the dashed line (— — —), and horizontal along the dashed-dotted line (—·—·—·). The isotropic fixed point is marked by ○, while the transverse fixed point is marked by □. Notice that all flows enter the isotropic fixed point tangential to the line $\bar\lambda_L^2=\bar\lambda_T^2$.

to the transverse fixed point. If $\bar\lambda_T=0$, the flow remains on the $\bar\lambda_L$ axis, but flows to large values of $\bar\lambda_L$—i.e., outside of the regime of validity of our $O(\epsilon)$ analysis. It should be noted that all flows approach the isotropic fixed point on paths asymptotically parallel to the line $\bar\lambda_L^2=\bar\lambda_T^2$.

In $d=2$, the isotropic and transverse fixed points collapse to the origin which attracts all flows. Now, for large enough l, any flow for which both $\bar\lambda^L$ and $\bar\lambda^T$ are both nonzero comes into the origin asymptotically parallel to the $\bar\lambda_L^2=\bar\lambda_T^2$ line, so that its long-wavelength behavior will be that of the isotropic case. The $\bar\lambda_T^2$ axis flows along itself into the origin and so has a different transverse long-wavelength behavior.

In both $d=2$ and $d=2-\epsilon$, we can estimate how large l must be, for a given initial condition $\bar\lambda_{L_0}\ll\bar\lambda_{T_0}$, in order that the fluid stop acting transversely and begin to appear isotropic. By examining (20c), it is clear that this will happen when $\bar\lambda_L\bar\lambda_T^2\sim\bar\lambda_T^3$ or $\bar\lambda_L\sim\bar\lambda_T$. For $\bar\lambda_L(l)\ll\bar\lambda_T(l)$, $\bar\lambda_T(l)$ satisfies

$$\frac{d\bar\lambda_T}{dl}=\tfrac{1}{2}\epsilon\bar\lambda_T-A_d\bar\lambda_T^3 \quad (23)$$

which integrates to

$$\bar\lambda_T(l)=\bar\lambda_{T_0}e^{\epsilon l/2}\left[1+2A_d\bar\lambda_{T_0}^2\frac{e^{\epsilon l}-1}{\epsilon}\right]^{-1/2} . \quad (24)$$

One then can notice that

$$\bar\lambda_L(l)=\bar\lambda_{L_0}\left[e^{\epsilon l/2}\frac{\bar\lambda_T(l)}{\bar\lambda_{T_0}}\right]^{1/2} , \quad (25)$$

which leads to

$$\bar\lambda_{T_0}/\bar\lambda_{L_0}^2\sim e^{\epsilon l_c/2}/(\epsilon)^{1/2} , \quad d<2$$
$$\bar\lambda_{T_0}/\bar\lambda_{L_0}^2\sim l_c^{1/2} , \quad d=2 . \quad (26)$$

We will later use the above result to estimate at what time a given fluid diffusive behavior $D_R(t),V_R(t)$ crosses over from transverse to isotropic.

The result that slightly compressible fluids behave at long wavelengths as if they had equal longitudinal and transverse velocity fluctuations can be understood on physical grounds. Any small amount of longitudinal fluctuations in $\vec{v}(\vec{r})$ will lead to a low density of traffic jams. Rescaling the theory just raises this density. But, diffusion in a fluid with a large density of such "slow regions" can be well modeled by a random walk between the randomly placed jams with random probabilities of jumping between jams. This is precisely the model which Luck[4] showed to correspond to the case of isotropic velocity fluctuations, $\bar\lambda_L=\bar\lambda_T\equiv\bar\lambda$.

The analysis of the model (6) near the isotropic fixed point has been done by Fisher,[5] and so we proceed to analyze the behavior for the transverse case. We first note that the scaling relations (18b) lead to the homogeneity relation for the renormalized diffusion constant (with $\bar\lambda_L=0$)

$$D_R(k,\omega,\bar\lambda_T)=\exp\left[2l-\int_0^l z(l')dl'\right]D_R\left[ke^l,\omega\exp\left[\int_0^l z(l')dl'\right],\bar\lambda_T(l)\right], \tag{27}$$

and for the moments displayed in Eq. (9),

$$\bar x_\mu(t,\vec v_0,\bar\lambda_T)=e^l\bar x_\mu\left[t\exp\left[-\int_0^l z(l')dl'\right],\vec v_0(l),\bar\lambda_T(l)\right], \tag{28}$$

$$\Delta\bar x_{\mu\nu}(t,\vec v_0,\bar\lambda_T)=e^{2l}\Delta\bar x_{\mu\nu}\left[t\exp\left[-\int_0^l z(l')dl'\right],\vec v_0(l),\bar\lambda_T(l)\right]. \tag{29}$$

We now can use the matching procedure of Ref. 2 to calculate the asymptotic properties of the above quantities as k, ω, and $\vec v_0$ go to zero, and as t goes to infinity.

For example, for small ϵ the perturbation theory for $\bar x_{\mu\nu}$ breaks down in the limit of large t and small v_0. It is not difficult to show that these divergences will be cut off when $l=l^*$ such that the condition

$$\frac{1}{[Dt(l^*)]^2}+\left[\frac{v_0(l^*)}{D}\right]^4=1, \tag{30}$$

where $t(l)=t\exp[-\int_0^l z(l')dl']$, is satisfied. For l large enough, $\bar\lambda_T(l)$ can be made arbitrarily small for $\epsilon\geq 0$, and arbitrarily close to $\bar\lambda_T^*=O(\epsilon^{1/2})$ for $\epsilon>0$. Consider the case $v_0=0$. Then, (30) implies $t(l^*)D\sim e^{-2l^*}tD=1$, or $e^{l^*}\sim(tD)^{1/2}$. For long times t, l^* will be large enough that a perturbative expansion of $\bar x_{\mu\nu}(t(l^*))$ will work, and we can set

$$\Delta\bar x_{\mu\nu}(t(l^*))=2Dt(l^*)\delta_{\mu\nu}[1+O(\bar\lambda_T^2(l^*))]. \tag{31}$$

The matching condition (30) implies that

$$e^{2l^*}=Dt\exp\left[\int_0^{l^*}\bar\lambda_T^2A_d dl\right]. \tag{32}$$

Inserting (31) and (32) back into (29) [after doing the integral in (32)] yields, in view of the definition (9c),

$$2tD_R(t)\delta_{\mu\nu}=2Dt\left[1+2A_d\bar\lambda_T^2\frac{e^{\epsilon l}-1}{\epsilon}\right]^{1/2}[\delta_{\mu\nu}+O(\bar\lambda_T^2(l^*))]. \tag{33}$$

We can now use (30) and (33) in conjunction to extract the long-time behavior by using the fact that $e^{\epsilon l^*}\simeq(Dt)^{\epsilon/2}$. For instance, if $\epsilon>0$ we have

$$D_R(t)\sim D(Dt)^{\epsilon/4}\text{ as }t\to\infty. \tag{34}$$

The crossover exponent from t dominated behavior to v_0 dominated behavior can also be extracted from (30) for the case of asymptotically small $1/t$ and v_0. It is helpful to rewrite (30) as

$$\frac{1}{(tD)^2}\exp\left[2\int_0^{l^*}\bar\lambda_T^2A_d dl\right]+\left[\frac{v_0}{D}\right]^4\times\exp\left[-4A_d\int_0^{l^*}\bar\lambda_T^2 dl\right]=e^{-4l^*}. \tag{35}$$

For $\epsilon\leq 0$ we see that, up to exponentially small corrections, the crossover occurs at $t\sim Dv_0^{-2}$. For $\epsilon>0$, replacing $\bar\lambda_T^2(l)$ by its fixed point value $\bar\lambda_T^{*2}=\epsilon/2A_d$ in (35) gives crossover at $t\sim Dv_0^{-2+\epsilon/2}$.

We can also use the matching procedure in conjunction with Eq. (26) to estimate the critical time t_c when an al-

TABLE I. Asymptotic forms for $D(k,\omega)$ and $\Delta\bar x_{\mu\mu}(t)$, and $\Delta\bar x_{\mu\mu}(t)/d$ valid in the transverse case $\bar\lambda_L=0$, in the limit k, ω, $1/t$ go to zero. Here D_R is the constant part of the renormalized diffusion constant for $d>2$. It can be shown that $D_R>D$.

	$D_R(k,0)$, $(v_0=0)$	$D_R(0,\omega)$, $(v_0=0)$	Crossover at
$\epsilon<0$	$=D_R[1+(\text{const})k^{\lvert\epsilon\rvert}]$	$=D_R[1+(\text{const})\omega^{\lvert\epsilon\rvert/2}]$	$\omega\sim Dk^2$
$\epsilon=0$	$\sim\left[\ln\frac{1}{k}\right]^{1/2}$	$\sim\left[\ln\frac{1}{\omega}\right]^{1/2}$	$\omega\sim Dk^2$
$\epsilon>0$	$\sim k^{-\epsilon/2}$	$\sim\omega^{-\epsilon/4}$	$\omega\sim Dk^2\left[\frac{k}{\Lambda}\right]^{-\epsilon/2}$
	$\Delta\bar x_{\mu\mu}(t)/d$, $t\ll\tau$	$\Delta\bar x_{\mu\mu}(t)/d$, $t\gg\tau$	$\tau(v_0)$
$\epsilon<0$	$=2D_Rt[1+(\text{const})t^{-\lvert\epsilon\rvert/2}]$	$=2D_Rt[1+(\text{const})v_0^{\lvert\epsilon\rvert}]$	$\tau\sim D/v_0^2$
$\epsilon=0$	$\sim t(\ln t)^{1/2}$	$\sim t\left[\ln\frac{1}{v_0}\right]^{1/2}$	$\tau\sim D/v_0^2$
$\epsilon>0$	$\sim t^{1+\epsilon/4}$	$\sim tv_0^{-\epsilon/2}$	$\tau\sim(D/v_0^2)(\Lambda D/v_0)^{\epsilon/2}$

most transverse fluid will cease to appear transverse. We assume $t_c \ll D v_0^{-2}$ for concreteness. Now $e^{l*} \sim (tD)^{1/2}$, so plugging into (26) yields

$$t_c \sim D^{-1} \exp(2\bar{\lambda}_{T_0}^2/\bar{\lambda}_{L_0}^4)\ , \quad d=2$$
$$t_c \sim D^{-1}\epsilon^{2/\epsilon}(\bar{\lambda}_{T_0}/\bar{\lambda}_{L_0}^2)^{4/\epsilon}\ , \quad d=2-\epsilon\ . \tag{36}$$

If one watched a real fluid, for which $0<\bar{\lambda}_L \ll \bar{\lambda}_T$, in two dimensions the diffusion would appear superdiffusive with $D_R(t) \sim (\ln t)^{1/2}$ for long times. Eventually this process will be cut off either by the presence of a nonzero v_0 when $t=\tau=Dv_0^{-2}$ or by the growth of $\bar{\lambda}_L(l^*)$ forcing the fluid to appear isotropic. In either case, the fluid will be forced back to diffusive behavior, $D(t) \sim D_R$. As τ has a power-law dependence on v_0 while t_c is exponential in $(\bar{\lambda}_T/\bar{\lambda}_L^2)^2$, expect that there are a large number of fluids and geometries for which $t \ll \tau \ll t_c$.

We have used the above techniques on all of the correlation functions defined by (8) and (9). Our results are summarized in Tables I and II. The properties of the transverse fixed point are summarized in Table I. In this case, we always find that $\bar{x}_\mu(t)=v_{0\mu}t$ with no corrections, as is discussed below. The isotropic correlation functions $D_R(t)$ for the case $v_0=0$ in Table II were calculated by Fisher.[5] We have used his $O(\epsilon^2)$ value of $z(l)$ valid near the isotropic fixed point

$$z=2+\frac{1}{4\pi^2}\bar{\lambda}^4 \tag{37}$$

in conjunction with our matching procedure to complete the correlation functions for the isotropic $\bar{\lambda}_L=\bar{\lambda}_T$ case.

The exponents in Table II for $\bar{x}^\mu(t)$ and $\Delta\bar{x}_{\mu\mu}(t)$ for $t \gg \tau$ and $\epsilon \geq 0$ are directly comparable to Luck's exponents. Our results for $\bar{x}^\mu(t)$ agree precisely with those in Luck's[6] formulas (50) and (51). However, all of our results for $\Delta\bar{x}_{\mu\mu}(t)$, i.e., the exponents which depend upon Fisher's[7] value of $1/2\pi^2$ in $z(l)=2+\bar{\lambda}^4/2\pi^2$, disagree with Luck. Luck finds $D_R(t) \sim D[1-4/\ln(1/v_0)]$ while we find $D_R(t) \sim D[1+2/\ln(1/v_0)]$ in $d=2$. Additionally, Luck finds $D_R(t) \sim v_0^{4\epsilon^2}$ for $t \gg \tau$ and $D_R(t) \sim t^{-2\epsilon^2}$ for $\tau \gg t$, while we find in agreement with Fisher[8] that $D_R(t) \sim v_0^{2\epsilon^2}$ for $t \gg \tau$ and $D_R(t) \sim t^{-\epsilon^2}$ for $\tau \gg t$.

It should be noted that for the incompressible case $(\bar{\lambda}_L=0)$, $\bar{x}^\mu(t,v_0)$ is unrenormalized. This result is actually true to all orders in ϵ. When $\bar{\lambda}_L=0$, the self-energy $\sum$ is explicitly of order k^2 for any finite l. Thus $\vec{v}_0$ is not graphically renormalized and

$$\vec{v}_0(l)=\vec{v}_0 \exp\left[\int_0^l (z-1)dl'\right]\ .$$

Using (28), we obtain

$$\bar{x}_\mu(t)=e^l v_0^\mu(l)t(l)=v_0^\mu t\ . \tag{38}$$

It is also interesting to examine the behavior of our model for totally compressible flows where $\lambda_T=0$. Now

$$\frac{d\bar{\lambda}_L}{dl}=\tfrac{1}{2}\epsilon\bar{\lambda}_L\ . \tag{39}$$

TABLE II. Asymptotic forms for $D(k,\omega)$, $\bar{x}^\mu(t)$, and $\bar{x}_{\mu\mu}(t)/d$ valid in the isotropic case $\bar{\lambda}_L=\bar{\lambda}_T=\bar{\lambda}$ in the limit, h, k, and $1/t$, go to zero. Here $v_R'^\mu$ and D_R' are the constant parts of the renormalized diffusion constant and velocity for $d \geq 2$. They can be shown to be functions of ϵ and $\bar{\lambda}_0$. One can further show that $D_R'<D$, and $v_R'<v_0$.

	$D_R(k,0)$, $(v_0=0)$	$D_R(0,\omega)$, $(v_0=0)$	Crossover at
$\epsilon<0$	$=D_R'[1+(\text{const})k^{\lvert\epsilon\rvert}]$	$=D_R'[1+(\text{const})\omega^{\lvert\epsilon\rvert/2}]$	$\omega \sim Dk^2$
$\epsilon=0$	$\sim D_R'\left[1+\frac{2}{\ln(1/k)}\right]$	$\sim D_R'\left[1+\frac{4}{\ln(1/\omega)}\right]$	$\omega \sim Dk^2$
$\epsilon>0$	$\sim Dk^{2\epsilon^2}$	$\sim D\omega^{\epsilon^2}$	$\omega \sim Dk^2\left[\frac{k}{\Lambda}\right]^{2\epsilon^2}$
	$\bar{x}^\mu(t)$, $t \ll \tau$	$\bar{x}^\mu(t)$, $t \gg \tau$	$\tau(v_0)$
$\epsilon<0$	$=tv_R'^\mu[1+(\text{const})t^{-\lvert\epsilon\rvert/2}]$	$=tv_R'^\mu[1+(\text{const})v_0^{\lvert\epsilon\rvert}]$	$\tau \sim D/v_0^2$
$\epsilon=0$	$\sim tv_0^\mu/\ln t$	$\sim tv_0^\mu/\ln(1/v_0)$	$\tau \sim D/v_0^2$
$\epsilon>0$	$\sim tv_0^\mu t^{-\epsilon/2}$	$\sim tv_0^\mu v_0^\epsilon$	$\tau \sim (D/v_0^2)\left[\frac{\Lambda D}{v_0}\right]^{2\epsilon}$
	$\Delta\bar{x}_{\mu\mu}(t)/d$, $t \ll \tau$	$\Delta\bar{x}_{\mu\mu}(t)/d$, $t \gg \tau$	$\tau(v_0)$
$\epsilon<0$	$=2D_R't[1+(\text{const})t^{-\lvert\epsilon\rvert/2}]$	$=2D_R't[1+(\text{const})v_0^{\lvert\epsilon\rvert}]$	$\tau \sim D/v_0^2$
$\epsilon=0$	$\sim 2D_R't\left[1+\frac{4}{\ln t}\right]$	$\sim 2D_R't\left[1+\frac{2}{\ln(1/v_0)}\right]$	$\tau \sim D/v_0^2$
$\epsilon>0$	$\sim Dt^{1-\epsilon^2}$	$Dtv_0^{2\epsilon^2}$	$\tau \sim (D/v_0^2)\left[\frac{\Lambda D}{v_0}\right]^{2\epsilon}$

For $d > 2$, $\bar{\lambda}_L$ flows to zero. Thus our previous matching procedure remains valid for this case. For $d = 2$, however, the $\bar{\lambda}_T = 0$ axis is an apparent fixed line, a result we do not believe to be true to higher order in ϵ. Below $d = 2$, the situation is even less well determined. Here, $\bar{\lambda}_L$ flows to large values, and so our matching scheme will break down, because we can no longer compute correlation functions at the matching point in perturbation theory. Because the leading longitudinal contribution to the renormalized diffusion constant is negative in perturbation theory, we believe this growth of $\bar{\lambda}_L(l)$ below two dimensions may be indicative of a localization transition. Just what the behavior actually is in the longitudinal case for $d \le 2$ remains an intriguing question.

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During the final stages of our investigation, we became aware of work by D. S. Fisher, D. Friedan, Z. Qiu, S. H. Shenker, and S. J. Shenker.[9] These authors have considered the generalized model defined by Eqs. (6) and (7) (in a different physical context, with $\vec{v}_0 = \vec{0}$), and have reached conclusions similar to ours. We are grateful for correspondence with D. S. Fisher which enabled us to uncover an error in our calculations for the intermediate case $\bar{\lambda}_T > \bar{\lambda}_L > 0$. We also benefited from correspondence with P. G. DeGennes, who pointed out interesting work by DeJong[10] on the anisotropic diffusion and logarithmic singularities which occur in the *large* $\vec{v}_0$ limit, which we have not considered here. One of us (J.A.A.) would like to acknowledge helpful discussions with S. Milner as well as a National Science Foundation graduate fellowship. This work was supported by the National Science Foundation through Grant No. DMR-82-07431.

APPENDIX: DISCUSSION OF RECURSION RELATIONS

To compute differential recursion relations, we use the procedure outlined in Ref. 2. As usual, we integrate out momenta between $e^{-\delta}$ and 1 and take the limit $\delta \to 0$. We also expand the graphs in k, v_0, and ω. Because $v_j(k)$'s rescaling factors $w^{L,T}(l) = d/2 + O(\epsilon)$ is positive and chosen to make the two-point cumulant marginal, the coefficients of higher-order cumulants of $\vec{v}(\vec{k})$ will be irrelevant. Similarly, because $z(l)$ has been chosen to make D marginal, a random contribution $\Delta(\vec{x})$ to the diffusion constant discussed in the Introduction will also be irrelevant at long wavelengths.

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