# Random walks in two-dimensional random environments with constrained drift forces

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Random walks in two-dimensional environments with a positionally random drift force are analyzed. If the force is constrained to be divergence-free, then the mean-square displacement is superdiffusive $\overline{\langle x^2(t)\rangle} \sim t\,(\ln t)^{1/2}$. If in addition the force has a component which is curl-free, there are two cases: If the two components are independent, the long-time behavior is diffusive with only logarithmic corrections; on the other hand, if the two components of the force are, respectively, parallel and perpendicular to the gradients of a single strongly fluctuating potential, the long-time behavior is subdiffusive and dominated by the longitudinal part.

Several authors[1–5] have recently considered the problem of a random walk in an environment in which there is a spatially random drift force $\mathbf{F}(\mathbf{x})$. Two dimensions was found to be the upper critical dimension for this problem:[3–5] for $d>2$, the long-time behavior is diffusive, while for $d<2$ the behavior is subdiffusive with the mean-square displacement growing as a power of the time

$$\overline{\langle x^2(t)\rangle} \sim t^{2/z} \tag{1}$$

with $z>2$ for $1<d<2$. (The bar denotes averaging over the walks and the brackets averaging over the random environments.) In the borderline case of two dimensions, the behavior is diffusive but there are universal logarithmic corrections at long times.[3]

In this paper, we will examine several related problems of random walks in random media in the interesting borderline dimension $d=2$. While at this stage it is unclear whether there are experimental systems which will show the interesting predicted behavior over some range of time scales, we will briefly discuss some possibilities at the end.

We consider a random walker with a drift force $\mathbf{F}(\mathbf{x})$ which satisfies a Langevin equation

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\eta}(t) + \mathbf{F}(\mathbf{x}) , \tag{2}$$

where $\boldsymbol{\eta}(t)$ is a Gaussian noise with correlations

$$\overline{\eta^\alpha(t)\eta^\beta(t')} = 2D\delta^{\alpha\beta}\delta(t-t') . \tag{3}$$

The associated Fokker-Planck equation for the probability distribution $\mathbf{P}(\mathbf{x},t)$ is

$$\frac{\partial P}{\partial t} = D\nabla^2 P - \nabla\cdot(\mathbf{F}P) . \tag{4}$$

Clearly, all the interesting physics will come from the behavior of the drift force and its correlations. We will in particular be concerned with the case of no uniform drift so that for all $\mathbf{x}$, $\langle \mathbf{F}(\mathbf{x})\rangle = \mathbf{0}$. In the earlier work,[3–5] $\mathbf{F}(\mathbf{x})$ was taken to be random with only short-range correlations and no constraints; we will call this *case A* defined by

$$\langle F^\alpha(\mathbf{x})F^\beta(\mathbf{x}')\rangle = \Delta\delta^{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}') . \tag{5}$$

Since we would expect physically that the curl-free and the divergence-free parts of a random force are likely to arise from different mechanisms, it is natural to divide up a more general force into a longitudinal curl-free part $\mathbf{E}$, and a transverse divergence-free part $\mathbf{B}$,

$$\mathbf{F} = \mathbf{E} + \mathbf{B} \tag{6}$$

with

$$\nabla\cdot\mathbf{B} = 0 \tag{7}$$

and

$$\nabla\times\mathbf{E} = \mathbf{0} . \tag{8}$$

The first new case we will consider, case B, is where $\mathbf{E}$ and $\mathbf{B}$ are independently distributed with different variances. Since the constraints Eq. (7) and (8) are long range, it is convenient to express the correlations in Fourier space as follows:

*Case B:*

$$\begin{aligned}\langle E^\alpha(\mathbf{q})E^\beta(\mathbf{q}')\rangle &= \Lambda\delta(\mathbf{q}+\mathbf{q}')\frac{q^\alpha q^\beta}{q^2} ,\\ \langle B^\alpha(\mathbf{q}')B^\beta(\mathbf{q}')\rangle &= T\delta(\mathbf{q}+\mathbf{q}')\left[\delta^{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right] ,\end{aligned} \tag{9}$$

and

$$\langle E^\alpha B^\beta\rangle = 0 , \tag{10}$$

where the dependences on the wave vectors $\mathbf{q}$ are just the longitudinal ($\Lambda$) and transverse ($T$) projection operators, respectively. As in the previous work, a short distance cutoff on the correlations is needed in order to make the problem well defined. The special case when the two forces have the same fluctuations $\Lambda = T \equiv \Delta$ is just case A.

An alternate way of interpreting the forces **E** and **B** in two dimensions is to consider them as being derived from the gradient and curl, respectively, of random potentials

$$E^\alpha = \nabla^\alpha V = (\nabla V)^\alpha \tag{11}$$

and

$$B^\alpha = \epsilon^{\alpha\beta}\nabla^\beta A \equiv (\nabla \times A)^\alpha . \tag{12}$$

In order to satisfy Eqs. (9) and (10), the potentials $V$ and $A$ must be independently distributed with momentum space variances of each proportional to $1/q^2$. Another natural possibility is that **B** and **E** arise from potentials which are the same up to some proportionality factor. The third case we will consider is the following.

*Case C* defined by Eqs. (11) and (12) with

$$A = \gamma V \tag{13}$$

and

$$\langle V(\mathbf{q})V(\mathbf{q}')\rangle = \frac{\Delta}{q^2}\delta(\mathbf{q}+\mathbf{q}') \tag{14}$$

so that

$$\langle F^\alpha(\mathbf{q})F^\beta(\mathbf{q}')\rangle = \Delta\frac{\delta(\mathbf{q}+\mathbf{q}')}{q^2}[q^\alpha q^\beta + \gamma(q^\alpha e^{\beta\gamma}q^\gamma + q^\beta \epsilon^{\alpha\gamma}q^\gamma) + \gamma^2(q^2\delta^{\alpha\beta} - q^\alpha q^\beta)] , \tag{15}$$

where the successive terms in Eq. (15) represent the $EE$ correlations, the cross $EB$ correlations implied by Eq. (13) and the $BB$ correlations. Note that because of the correlations between the longitudinal and transverse parts of **F**, case C never reduces to case A; these cross correlations are the only difference between cases B and C. Thus for $\gamma = 0$, case C reduces to case B with $T = 0$ and $\Lambda = \Delta$ and for $\gamma \to \infty$ it reduces to case B with $\Lambda = 0$ and $T = \gamma^2\Delta$.

We are interested in the long-time behavior of a particle which started at the origin at time zero, in particular its mean-square displacement which is defined by

$$\langle \overline{x^2(t)}\rangle = \int d^dx \, |\mathbf{x}|^2 \langle P(\mathbf{x},t)\rangle \tag{16}$$

with $P(\mathbf{x},t)$ the probability that a walker which starts at the origin at time zero will be at $\mathbf{x}$ at time $t$. In the limit of weak drift forces, we can perform a perturbative renormalization group analysis about the pure random walk with $\mathbf{F}=0$. This can be straightforwardly done by the Martin-Siggia-Rose—De Dominicis[6,7] functional-integral methods used in Ref. 3 for case A. The details will not be discussed here. We will perform a differential rescaling of lengths by an amount $e^l$. In order to keep the diffusion constant equal to its bare value (which we may take to be 1) we must rescale times by $e^{zl}$. At the pure diffusive fixed point with $\mathbf{F}=\mathbf{0}$, $z=2$. It is straightforward to show that the only new terms which can be generated by the renormalization-group transformation (such as four-point correlations in the $F$'s) are irrelevant at the pure diffusive fixed point about which we are expanding.

For case B, the differential recursion relations truncated at second order in the force correlations are

$$\frac{d\Lambda}{dl} = (2-d)\Lambda + \frac{\Lambda(\Lambda-T)}{2\pi} - \frac{\Lambda^2}{2\pi} + \frac{\Lambda T}{4\pi} , \tag{17}$$

$$\frac{dT}{dl} = (2-d)T + \frac{T(\Lambda-T)}{2\pi} - \frac{\Lambda T}{4\pi} , \tag{18}$$

and the dynamic rescaling is determined by

$$z = 2 + \frac{\Lambda - T}{4\pi} + O(\Lambda^2, \Lambda T, T^2) . \tag{19}$$

The first terms in Eqs. (17) and (18) express the marginality in two dimensions; we will henceforth work in $d=2$ and drop these terms. The second terms in Eqs. (17) and (18) arise from the temporal rescaling given by Eq. (19). Note that there are no terms in the recursion relation for $T$ of the form $\Lambda^2$ and visa versa. This suggests that both a purely transverse and a purely longitudinal force remain that way under renormalization. This can be seen to be a general consequence of reflection symmetry. We first analyze the consequences of the flow equations (17) and (18) for the extreme limits of a purely divergence-free and a purely curl-free force.

(i) Purely transverse force: $E=0$. If the force is divergence-free initially, then because of reflection symmetry it will remain so under renormalization. We thus simply have the recursion relations (in $d=2$)

$$\frac{dT}{dl} = -\frac{T^2}{2\pi} \tag{20}$$

with the dynamic rescaling determined by

$$z = 2 - \frac{T}{4\pi} . \tag{21}$$

The pure diffusive fixed point at $T=0$ is thus marginally stable. The mean-square displacement of a walker at long times $t_0$ can be found by integrating the recursion relations until a scale $l^*$ at which the renormalized time

$$t(l^*) = \exp\left[-\int_0^{l^*} z(l')dl'\right] t_0 \tag{22}$$

is of order 1. (We will use subscripts zero to denote unrenormalized values.) At this time scale the mean-square displacement $\langle \overline{x^2(l^*)}\rangle$ will be of order 1 (i.e., of the order of the length cutoff in the correlations of $F$), since the disorder will be weak but the diffusion constant still 1. The mean-square displacement $\langle \overline{x_0^2(t_0)}\rangle$ at time $t_0$ is then obtained simply by using $x_0 = e^{-l^*}x(l^*)$. From Eqs. (20) and (21), it is found that for the divergence-free case the behavior is superdiffusive with

$$\langle \overline{x^2(t)}\rangle \sim t(\ln t)^{1/2} . \tag{23}$$

The diverging logarithmic factor comes from the slow transient given by Eq. (20) and the integrated effect of the faster than diffusive time rescaling given by Eq. (21).

Physically, the superdiffusive behavior is caused by the effect of the large loops of lines of force along which the walker can travel semiballistically for long times. This effect is somewhat analogous to the logarithmically infinite self-diffusion coefficient of a particle in a two-dimensional fluid which is caused by the effect of large-scale vorticity.[8] Mathematically, the sign of the correction to the diffusive behavior Eq. (23) can be shown to arise from the *anti*-Hermitian nature of the perturbation to the Fokker-Planck equation (4) in the divergence-free case. This perturbation *raises* the low eigenvalues of the Fokker-Planck operator at the first nontrivial order in perturbation theory.

We now turn to the other limit of a purely curl-free force.

(ii) Longitudinal force: $B=0$. In this case the second-order terms in Eq. (17) exactly cancel and we are left with

$$\frac{d\Lambda}{dl}=0+O(\Lambda^3) \tag{24}$$

and

$$z=2+\frac{\Lambda}{4\pi} . \tag{25}$$

The sign of the correction term in the temporal rescaling is *opposite* to that for the transverse case. This is caused by regions of the system in which $\nabla\cdot\mathbf{F}$ is negative which tend to temporarily trap walkers and slow down diffusion. Unfortunately, due to the absence of a leading term in Eq. (24), we would need to calculate the term of order $\Lambda^3$ to determine the long-time behavior in $d=2$; although from the sign of $z-2$ we know generally that the behavior will be *subdiffusive.* There are three possibilities:

(a) there is a *negative* term of order $\Lambda^n$ (with $n\geq 3$) in the recursion relation for $\Lambda$,

(b) there is a *positive* term of order $\Lambda^n$, or

(c) $d\Lambda/dl$ is zero to all orders in $\Lambda$.

If (a) is correct, which is most likely to be true, then the mean-square displacement would behave as

$$\langle \overline{x^2(t)} \rangle \sim t\exp[-c(\ln t)^{(n-2)/(n-1)}] \tag{26}$$

with $c$ a universal constant. This behavior is intermediate between a diffusive and a subdiffusive power law. If (b) is correct, on the other hand, the long-time behavior in two dimensions would be controlled by a strong disorder fixed point which would presumably exhibit a subdiffusive power-law growth of $\langle \overline{x^2(t)} \rangle$. In $2+\epsilon$ dimensions the existence of a positive higher-order term in $d\Lambda/dl$ would imply a critical fixed point with $\Lambda^*\sim\epsilon^{1/(n-1)}$ which would separate a diffusive regime for small $\Lambda$ from a subdiffusive one for large $\Lambda$. It might be possible, however, to exclude this behavior on general grounds by trying to show that (for nonpathological distributions of the disorder) the behavior for $d>2$ will always be diffusive.[9] The third possibility, (c), would suggest, at least naively, that in $d=2$ there could be subdiffusive power-law growth of $\langle \overline{x^2(t)} \rangle$ with a continuously variable power law.

We now consider the *general case B* with both $\Lambda$ and $T$ nonzero.

For $d=2$ we have to leading nontrivial order

$$\frac{d\Lambda}{dl}=-\frac{\Lambda T}{4\pi} \tag{27}$$

and

$$\frac{dT}{dl}=-\frac{T^2}{2\pi}+\frac{\Lambda T}{4\pi} . \tag{28}$$

These flow equations have a constant of the motion

$$K=\frac{\Lambda-T}{\Lambda^2} \tag{29}$$

in terms of which

$$\frac{d\Lambda}{dl}=-\Lambda^2+K\Lambda^3 \tag{30}$$

and

$$z=2+\frac{K\Lambda^2}{4\pi} . \tag{31}$$

From Eq. (30) it can be seen that $\Lambda(l)$ will tend to zero as $l^{-1}$ for large $l$ implying that $\Lambda-T$ will go to zero as $l^{-2}$. The flows from Eqs. (27) and (28) are sketched in Fig. 1. From integrating the $z$ equation (29), it can be seen that for any finite $K$, the renormalization of the time will only give rise to logarithmic *corrections* to the diffusion, i.e.,

$$\langle \overline{x^2(t)} \rangle \sim t\left[1+\frac{C}{\ln t}\right] . \tag{32}$$

In contrast to the random force case A which yielded Eq. (32) with $C$ a universal constant equal to 4,[3] the constant in the general case B is nonuniversal. It will tend to be large for weak disorder, positive if the transverse part of the force is larger than the longitudinal part, and negative if the longitudinal part is larger. We thus see that, at long length scales, the behavior of the system with independent transverse and longitudinal parts of the force will tend to look like a totally random force as in case A. We must note here, however, that the behavior for $\Lambda$ small and $T/\Lambda$ small but nonzero is unclear since if a positive higher-order term of the form $\Lambda^n$ existed in the $d\Lambda/dl$ equation [possibility (b) discussed above] then the flows could go out of the perturbative regime before $T$ became large enough to dominate the renormalization of $\Lambda$. How-

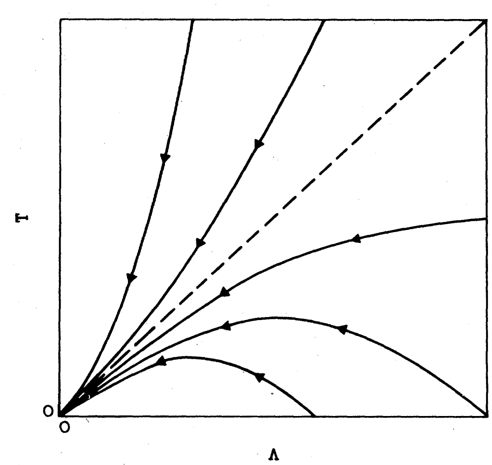


FIG. 1. Renormalization-group flows for weak disorder for general case B where $\Lambda$ and $T$ are, respectively, the variances of the longitudinal and transverse parts of the random drift forces. Behavior on the line $T=0$ is uncertain due to possible higher-order terms.

ever, for any *fixed* nonzero ratio of Λ to $T$, the behavior for sufficiently weak disorder will be controlled by the truncated recursion relations Eqs. (27) and (28).

We now turn to *case C* where **B** and **E** are related to the curl and gradient of the *same* potential. This property is preserved by renormalization at least at second order. The recursion relations for this case are found to be

$$\frac{d\Delta}{dl}=(2-d)\Delta+0\Delta^2+O(\Delta^3) \tag{33}$$

and

$$\frac{d\gamma^2}{dl}=\Delta\left[-\frac{\gamma^2}{2\pi}-\frac{\gamma^4}{2\pi}\right]+O(\Delta^2) \tag{34}$$

and the temporal rescaling is

$$z-2=\frac{\Delta}{4\pi}(1-\gamma^2)+O(\Delta^2) \tag{35}$$

which we note is the same at this order as for case B where the correlations between **B** and **E** did not exist. This is just a consequence of reflection symmetry. For $\gamma\rightarrow 0$ or $\infty$, the recursion relations Eqs. (33) and (34) for $\Delta$ or $\gamma^2\Delta$, respectively, reduce to those for case B for pure Λ [Eq. (24)] or pure $T$ [Eq. (20)], respectively.

From Eq. (34) we see that for any finite $\gamma$, $\gamma$ will decrease towards zero as shown in the flows sketched in Fig. 2 and the long-time behavior will be dominated by the longitudinal part of the force. As for the purely longitudinal limit of case B, the very long-time behavior in $d=2$ will be controlled by the higher-order terms in Eq. (33) which we have not calculated. Unfortunately the superdiffusive behavior for the purely transverse case which comes from the walker moving around large constant potential contours will be observable only for a finite range of times in the presence of even a small longitudinal component of the force. If initially $\Delta_0\gamma_0^2\sim 1$ but $\Delta_0 \ll 1$, then

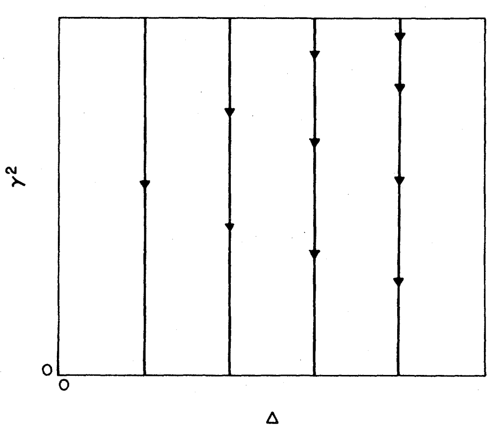


FIG. 2. Renormalization-group flows for weak disorder for case C. The random force has components with relative magnitude $\gamma$, perpendicular and parallel to the gradient of a random potential which has variance in Fourier space, $\Delta/q^2$. Flows are more rapid on the right and upper parts of the diagram. Behavior on the $\gamma^2=0$ line which controls the asymptotic behavior is uncertain.

for a range of length scales up to a scale of order $e^{2\pi/\Delta_0}$, the behavior will be dominated by the transverse force, while at longer scales it will be dominated by the longitudinal force. The domination of the long-time behavior by the longitudinal force **E** arises from the effects of **E** tending to prevent the walker from remaining near the large constant potential contours along which it would move in the absence of **E**. The deep wells in the potential then trap the walker and make the long-time behavior subdiffusive.

We have seen that a variety of interesting diffusive, superdiffusive, and subdiffusive behavior is possible for two-dimensional random walks subject to positionally random forces. We now briefly discuss possible applications.

An interesting application of case B has been suggested by Aronowitz and Nelson,[10] who consider also the effects of a small *uniform* drift which acts as a low-frequency cutoff. They analyze the diffusion of particles in a fluid in steady-state flow through a random porous medium. The drift forces **B** and **E** represent the transverse, $\mathbf{v}_T$, and longitudinal, $\mathbf{v}_L$, random parts of the local fluid velocity. The intermediate-time behavior in the limit that the average flow velocity is much less than the fluctuations in $\mathbf{v}_T$ and $\mathbf{v}_L$ is governed by case B.

For case C, since the drift force arises from a *single* potential $V$, it is natural to consider systems with a strongly fluctuating potential. The most natural example of this is motion on the surface of a liquid with the potential depending on the height. Out to distances of the order of the capillary length, the fluctuations in the height have mean-square Fourier magnitudes proportional to $1/q^2$. On time scales shorter than the capillary wave motion, a liquid will appear as a static potential. Some deposited amorphous films which are static on very long-time scales also appear to have thickness fluctuations which have the same $1/q^2$ spectrum[11] and thus should be interesting systems to study. The motion of a light particle on the surface of such a frozen film in the presence of thermal noise will undergo motion described roughly by case B with a purely longitudinal force. The long-time behavior will thus be subdiffusive. In the presence of a large magnetic field **H** perpendicular to the film, the Lorentz forces on a charged particle will tend to give the particle a drift perpendicular to **H** and the surface electric field which is naturally proportional to $\nabla V$. Thus there is a drift in the direction of $\nabla\times V$. If this transverse drift dominates the longitudinal drift down the potential, then for a range of times, it might be possible to observe the superdiffusive behavior associated with the purely transverse case. Unfortunately, at sufficiently long times the longitudinal force will always make the behavior subdiffusive since this is just the general case C. The superdiffusive regime may thus be hard to observe.

Another example for which transverse components of the force exist is for flux flow of vortices in type-II superconductors. The conservative Magnus force tends to make the vortices move transverse to potential gradients while dissipative effects make them move down the potential gradients.[12] The ratio $\gamma_0$ between the transverse and longitudinal drifts tends to be very small,[13] however it is possible that in some granular systems, the opposite limit may apply, with $\gamma_0 \gg 1$.[14] If this were the case and vortex interactions could be ignored, there might be a regime of subdiffusive behavior.

Unfortunately, for both the charged particle in a large magnetic field and for the vortices, the motion down the potential is related to the microscopic diffusion constant by the fluctuation-dissipation theorem. Thus it is likely to be difficult to find regimes where the microscopic diffusion is appreciable but the flow down the potential negligible.

Finally, examples of disorder with long-range correlations can be found in systems near second-order phase transitions. Critical slowing down can, depending on the value of the dynamic exponents,[15] freeze the fluctuations over long enough time scales to be effectively static at long wavelengths. Fluctuations in the density of a fluid at its critical point should affect the diffusion of an impurity at long times. The correlations will unfortunately fall off somewhat more rapidly at long distances from the marginal $1/q^2$ case in two dimensions discussed here, except perhaps in the presence of long-range interactions.

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