Critical Bernoulli Site Percolation on $\mathbb{Z}^d$

A block-renormalization argument

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Scope and verification status. The mathematical argument in this manuscript treats every integer dimension $d \geq 2$. A separate Lean 4 development proves the final criticality theorem for $d \geq 3$; it does not currently cover the manuscript's $d = 2$ case. This is a rough document and has not yet received much human review.

Mathematics generated by ChatGPT 5.6, prompted by Ahmed Bou-Rabee, with exposition and formalization support by Claude Opus 5.

Abstract

We prove that nearest-neighbor Bernoulli site percolation on every integer lattice has no infinite cluster at its critical parameter. The finite input is a multiplicative gluing inequality for independent Bernoulli hyperedges. Site percolation becomes such a model by replacing each open site with one open hyperedge joining the auxiliary vertices on its incident edges. A technical issue is that two different hyperedges can reach the same boundary vertex, so ordinary boundary sets do not preserve intersections in the induction. We retain the hyperedge names as source labels and attach a vertex set to each label. Union and intersection are then exact at the label level, while connectivity still uses the union of the attached vertex sets. This gives the conditioned cluster inequalities needed for multiplicative gluing. A site version of the Kozma–Nitzan block construction gives a finite adaptive block-exploration rule whose steps read at most M fresh sites and have a uniform success margin. We fix the scales and rule selected at a percolating parameter p and sample the fresh sites at a slightly smaller parameter $q < p$. Coupling the two product measures with the same independent uniforms changes the outcome of a step with probability at most $M(p - q)$. The resulting process still stochastically dominates a supercritical planar site process. Thus every percolating parameter has a smaller percolating parameter, which rules out percolation at the critical parameter directly, without a half-space or planar-criticality input.

Contents

1 The result

Let each vertex of $\mathbb{Z}^d$ be open independently with probability p. An open path is a nearest-neighbor path all of whose vertices are open. We write $C(x)$ for the open cluster of x, with $C(x) = \varnothing$ when x is closed. Define

$$\theta_{\text{site}}(p) = \mathbb{P}_p(C(0) \text{ is infinite}), \qquad p_{\text{c}}^{\text{site}} = \inf\{p : \theta_{\text{site}}(p) > 0\}.$$

Theorem 1.1 (Critical site percolation)**.** *For every integer $d \geq 2$,*

$$\theta_{\text{site}}(p_{\text{c}}^{\text{site}}) = 0.$$

The proof has two parts. First, Theorem 5.4 proves a finite gluing inequality for independent hyperedges. Corollary 6.1 transfers it to site percolation when the named terminals are forced open. Second, Proposition 8.13 inserts that corollary into a site-based block exploration and then fixes the resulting finite examination rule. Uniform comparison of bounded decision trees shows that if site percolation occurs at p, it already occurs at some $q < p$. This conclusion contradicts the definition of the critical parameter.

The requirement that named terminals are open is natural and necessary. Without it, $\mathbb{P}(o \leftrightarrow A) \leq p_o$, so a near-one hypothesis is impossible when $p_o < 1$. In the lattice argument the source, target, and intermediate vertices arise by wiring or by conditioning and are therefore open with probability one.

2 Independent hyperedges

We first describe the finite model. Let V be a finite vertex set, and let E be a finite set of labels. Each $e \in E$ has an incidence set $\iota(e) \subseteq V$ containing at least two vertices. Independently for each e, declare e open with probability $p_e \in [0,1]$. Two vertices are connected when a chain of open hyperedges joins them. Connectivity is reflexive in this finite hypergraph model. For $S \subseteq V$, let C_S be the union of the connected components that meet S.

For conditional arguments we retain slightly more information. Define

$$\mathcal{K}_S = \{e \in E : e \text{ is open and } \iota(e) \cap C_S \neq \varnothing\}.$$

Its support is C_S, with the roots S retained even when no incident hyperedge is open. If $x \in S$, then C_x is recoverable from $\mathcal{K}_S$ by taking the component of x in these open hyperedges. We order the possible values of $\mathcal{K}_S$ by inclusion. Their supports and each rooted component increase in this order.

For $U \subseteq V$, the induced hypergraph on U retains precisely the hyperedges whose incidence sets lie in U. If $K \subseteq U$, deleting K means passing to the induced hypergraph on $U \setminus K$; thus every hyperedge meeting K is removed. We use a superscript U when the ambient vertex set matters.

2.1 Site representation

The following exact construction explains why hyperedges are the right finite model. It also explains why replacing a site by independent incident edges would be incorrect.

Lemma 2.1 (Site-to-hyperedge representation)**.** *Let $G = (V, E_G)$ be a finite graph with independent site probabilities $(p_v)_{v \in V}$. There is an independent hyperedge model such that, for all distinct vertices $x, y \in V$, the event $\{x \leftrightarrow y\}$ is identical in the two models. The identity also holds for $x = y$ when $p_x = 1$.*

Proof. For each graph edge $uv \in E_G$, introduce a new auxiliary vertex m_{uv}. For each non-isolated site v, introduce the hyperedge

$$H_v = \{v\} \cup \{m_{uv} : uv \in E_G\}.$$

Declare H_v open with probability p_v, independently over v. No hyperedge is needed for an isolated site because it cannot participate in a connection between distinct vertices.

Adjacent sites u and v have intersecting hyperedges because $m_{uv} \in H_u \cap H_v$. It follows that an open site path gives a chain of open hyperedges. Conversely, consecutive hyperedges in a chain can intersect only at an auxiliary vertex m_{uv} belonging to adjacent sites u and v. Removing the auxiliary vertices from the chain gives an open site path. This proves the event identity for distinct terminals. If $x = y$ and $p_x = 1$, then both models regard x as connected to itself. $\square$

2.2 Cluster exploration

Let $Z \subseteq U$. Write

$$E_Z = \{e \in E : \iota(e) \cap Z \neq \varnothing\}$$

for the hyperedges incident to Z. If $I \subseteq E_Z$ is the set of open incident hyperedges, define its trace outside Z by

$$T_Z(I) = \bigcup_{e \in I} (\iota(e) \setminus Z). \tag{1}$$

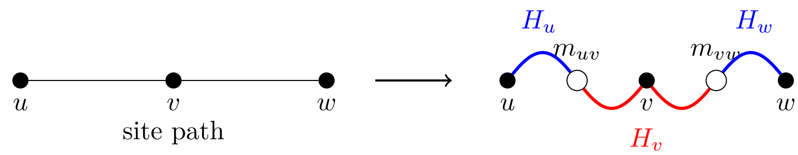

Figure 1: An open site activates one hyperedge containing the site and the auxiliary vertices on its incident edges. Adjacent open sites communicate through their common auxiliary vertex. The colors distinguish labels; equal colors do not indicate a shared random variable.

The trace obeys

$$T_Z(I \cup J) = T_Z(I) \cup T_Z(J), \qquad T_Z(I \cap J) \subseteq T_Z(I) \cap T_Z(J). \tag{2}$$

The second relation need not be equality: two different hyperedges can reach the same vertex.

Lemma 2.2 (Exploration identity)**.** *Let $N, Z \subseteq U$ with $Z \subseteq N$. Condition on the states of the hyperedges in E_Z, and let $I \subseteq E_Z$ be the open set. In every event on which a cluster from $U \setminus Z$ avoids N, deleting Z replaces the source N by*

$$(N \setminus Z) \cup T_Z(I).$$

The hyperedges disjoint from Z retain their original independent laws.

Proof. Every path that first enters Z uses an open member of E_Z. Its last vertex before leaving Z lies in the trace in (1). Conversely, each vertex in the trace is joined to Z by its defining open hyperedge. After the incident hyperedges have been exposed, the remaining coordinates are disjoint from E_Z and therefore keep their product law. These two observations give the claimed source replacement and the conditional law. □

We also use exact conditioning on a cluster. If $C_S = K$, then each hyperedge meeting both K and $U \setminus K$ is closed. Hyperedges disjoint from K remain independent. Thus the unexplored configuration is precisely the original product model on the induced hypergraph $U \setminus K$.

3 Cluster inequalities

The finite gluing proof needs two forms of conditional association. We give the hyperedge proofs because the graph versions alone do not cover site percolation. We first record finite-product versions of the two elementary inequalities used in the induction.

Lemma 3.1 (Finite Harris inequality)**.** *Let $(X_e)_{e \in E}$ be finitely many independent Bernoulli variables. If f, g are increasing real functions on $\{0,1\}^E$, then*

$$\mathrm{Cov}(f, g) \geq 0.$$

If one function is increasing and the other decreasing, the covariance is nonpositive. Two decreasing functions have nonnegative covariance.

Proof. Induct on $|E|$. Deterministic coordinates may be removed. For a remaining coordinate e, write $q = \mathbb{P}(X_e = 1) \in (0,1)$ and let $m_f(b) = \mathbb{E}[f \mid X_e = b]$, with the analogous notation for g.

Conditional on $X_e = b$, the induction hypothesis applies to the remaining product coordinates. Moreover $m_f(1) \geq m_f(0)$ and $m_g(1) \geq m_g(0)$. The law of total covariance gives

$$\operatorname{Cov}(f,g) = \sum_{b=0}^{1} \mathbb{P}(X_e = b) \operatorname{Cov}(f, g \mid X_e = b) + q(1-q)(m_f(1) - m_f(0))(m_g(1) - m_g(0)) \geq 0.$$

The other sign statements follow by replacing a decreasing function by its negative. $\square$

Lemma 3.2 (Two-point summation)**.** *Let $a_i, b_i, c_i, d_i \geq 0$ for $i \in \{0,1\}$, and suppose*

$$a_i b_j \leq c_{\max\{i,j\}} d_{\min\{i,j\}} \qquad (i, j \in \{0,1\}).$$

Then

$$(a_0 + a_1)(b_0 + b_1) \leq (c_0 + c_1)(d_0 + d_1).$$

Proof. The two diagonal terms are bounded directly. For the cross terms put

$$x = a_0 b_1, \quad y = a_1 b_0, \quad A = c_1 d_0, \quad B = c_0 d_1.$$

Then $x, y \leq A$ and

$$xy = (a_0 b_0)(a_1 b_1) \leq (c_0 d_0)(c_1 d_1) = AB.$$

We claim $x + y \leq A + B$. If $B \geq A$, this follows from $x, y \leq A$. Suppose $B < A$. If $x \leq B$, use $y \leq A$. If $B < x \leq A$, then $y \leq AB/x$ and

$$A + B - x - \frac{AB}{x} = \frac{(A-x)(x-B)}{x} \geq 0.$$

The case $x = 0$ is immediate. Adding the diagonal bounds proves the claim. $\square$

Theorem 3.3 (Finite four-functions theorem)**.** *Let E be finite and let $\alpha, \beta, \gamma, \delta : 2^E \to [0, \infty)$. If*

$$\alpha(I)\beta(J) \leq \gamma(I \cup J)\delta(I \cap J) \qquad (I, J \subseteq E),$$

then

$$\left(\sum_{I \subseteq E} \alpha(I)\right)\left(\sum_{J \subseteq E} \beta(J)\right) \leq \left(\sum_{K \subseteq E} \gamma(K)\right)\left(\sum_{L \subseteq E} \delta(L)\right).$$

Proof. Induct on $|E|$. The empty case is the hypothesis itself. Choose $e \in E$, put $E' = E \setminus \{e\}$, and split each function according to whether e is absent or present. For $i, j \in \{0,1\}$, the induction hypothesis applied on E' to the four slices $\alpha_i, \beta_j, \gamma_{\max\{i,j\}}, \delta_{\min\{i,j\}}$ gives

$$A_i B_j \leq C_{\max\{i,j\}} D_{\min\{i,j\}},$$

where A_i is the sum of the ith slice of α, and similarly for the other letters. Lemma 3.2 gives $(A_0 + A_1)(B_0 + B_1) \leq (C_0 + C_1)(D_0 + D_1)$, which is the required inequality. $\square$

For an increasing nonnegative function f of a vertex cluster, write

$$K_U(f; X) = \mathbb{E}_U\left[f(C_s); C_s \cap X = \varnothing\right].$$

Lemma 3.4 (Conditional association after one cluster exposure)**.** *Let U be a finite vertex set in an independent hyperedge model. Let $s \in U$, let $X, Y \subseteq U$, and let f, g be increasing nonnegative functions of subsets of U. Then*

$$K_U(f; X)K_U(g; Y) \leq K_U(fg; X \cap Y)K_U(1; X \cup Y). \tag{3}$$

Consequently, conditional on $C_s \cap X = \varnothing$, increasing functions of C_s are positively associated.

Proof. We prove (3) by strong induction on $|U|$. Cases with $s \in X \cup Y$ have a zero factor. Let $Z = X \cap Y$.

Suppose first that $Z = \varnothing$. The maps $f(C_s)$ and $g(C_s)$ are increasing in the hyperedge coordinates, whereas both avoidance indicators are decreasing. Two applications of Harris' inequality, followed by Harris' inequality for $f(C_s), g(C_s)$ and for the two avoidance indicators, give (3).

For the inductive step, suppose that $Z \neq \varnothing$. Expose the states $I \subseteq E_Z$ and delete Z. Lemma 2.2 writes the four factors in (3) as sums over I. For two exposed states $I, J \subseteq E_Z$, apply the induction hypothesis in $U \setminus Z$ with the avoided sets

$$(X \setminus Z) \cup T_Z(I) \quad \text{and} \quad (Y \setminus Z) \cup T_Z(J).$$

The union of these sets equals

$$((X \cup Y) \setminus Z) \cup T_Z(I \cup J).$$

Their intersection contains $T_Z(I \cap J)$ by (2). Avoidance probabilities decrease when the avoided set grows. The induction hypothesis therefore gives the pointwise four-functions inequality indexed by $I, J, I \cap J, I \cup J$.

Let $\pi(I)$ denote the product probability of the exposed state I. The identity

$$\pi(I)\pi(J) = \pi(I \cap J)\pi(I \cup J)$$

turns that pointwise inequality into the hypothesis of the finite four-functions theorem, Theorem 3.3. Summing over I and J proves (3). This completes the induction. $\square$

The proof also gives a version for functions of $\mathcal{K}_s$. The functions f and g may be increasing functions of $\mathcal{K}_s$, rather than only of its support. In the inductive step, whenever the cluster avoids the exposed set Z, none of the open hyperedges incident to Z belongs to $\mathcal{K}_s$. Thus $\mathcal{K}_s$ is exactly the same set when computed in the induced model on $U \setminus Z$, and the four-functions comparison is unchanged. The same statement holds for a finite source set: adjoin a deterministic hyperedge from a new root to that set and apply the singleton statement.

The next result concerns $(\mathcal{K}_S, \mathcal{K}_T)$. A function of this pair is called increasing–decreasing when it is increasing in the first coordinate and decreasing in the second.

Lemma 3.5 (Conditional association after two cluster exposures)**.** *Let $S, T \subseteq U$ be disjoint finite source sets in an independent hyperedge model. Conditional on $C_S \cap T = \varnothing$, two nonnegative increasing–decreasing functions of $(\mathcal{K}_S, \mathcal{K}_T)$ are positively associated.*

Proof. Condition first on $\mathcal{K}_S = J$, and let $K = C_S$ be its support. Every hyperedge meeting both K and its complement is then closed, while the hyperedges disjoint from K remain independent. Conditional on J, both functions decrease as the remaining coordinates enlarge $\mathcal{K}_T$. Harris' inequality gives positive association within this conditional law.

If $J \subseteq J_1$, then the corresponding support and the family of deleted hyperedges grow. Couple the two residual models with the same coordinate variables. The record $\mathcal{K}_T$ can only decrease. Hence the two conditional means increase with J: their first argument grows and their second argument shrinks. The version of Lemma 3.4, applied to the law of $\mathcal{K}_S$ conditioned to avoid T, positively associates these means. Averaging the conditional Harris inequality proves the claim. This is the hyperedge version of the two-cluster conditional association argument of van den Berg, Häggström, and Kahn [2]. □

4 Retaining hyperedge and source labels

The induction needed for gluing exposes all hyperedges incident to a common source. Recording only the resulting vertex trace is insufficient because the second relation in (2) may be strict. We retain the names of the exposed hyperedges and retain their vertex incidence sets.

Let L be a finite label set, and let $\alpha : L \to 2^U$ assign an incidence set to each label. For $Q \subseteq L$ and $W \subseteq U$, write

$$[Q]_W = W \cap \bigcup_{\ell \in Q} \alpha(\ell).$$

Thus Q denotes the ordinary vertex source $[Q]_W$ in the hypergraph induced on W. Different labels may have equal or overlapping incidence sets. Retaining distinct labels when their incidence sets overlap is precisely what preserves intersections.

We use the following convention in induced hypergraphs. If a named vertex does not belong to W, then its cluster in W is empty and a connection to that vertex is false. Fix vertices $x, o, v \in U$, a set $A \subseteq U$, and a function $F : 2^U \to [0, \infty)$. For $Q \subseteq L$, let $R_Q^W = W \setminus C^W_{[Q]_W}$ and define

$$E_A^W(Q) = \mathbb{P}_W(o \leftrightarrow v,\ v \not\leftrightarrow A \cup [Q]_W), \tag{4}$$

$$M_A^W(Q) = \mathbb{P}_W(v \not\leftrightarrow A \cup [Q]_W), \tag{5}$$

$$Y_F^W(Q) = \mathbb{E}_W[F(R_Q^W); x \notin C^W_{[Q]_W}]. \tag{6}$$

The same notation with a vertex set $B \subseteq W$ in place of Q means that the source is B. Let

$$q_A(W) = \begin{cases} E_A^W(\varnothing)/M_A^W(\varnothing), & M_A^W(\varnothing) > 0, \\ 0, & M_A^W(\varnothing) = 0, \end{cases}$$

and define

$$X_F^W(Q) = \mathbb{E}_W[q_A(R_Q^W)F(R_Q^W); x \notin C^W_{[Q]_W}]. \tag{7}$$

The next lemma supplies the one-source estimate required by the labeled induction. Its proof is included because the stopping argument is unchanged when graph edges are replaced by hyperedges.

Lemma 4.1 (Decision-tree Harris inequality)**.** *Let a finite family of Bernoulli coordinates be independent, let $\mathcal{T}$ be a finite adaptive decision tree, and let $\mathcal{F}_\mathcal{T}$ be its terminal transcript. If f and g are increasing functions of the coordinates, then*

$$\mathrm{Cov}(\mathbb{E}[f \mid \mathcal{F}_\mathcal{T}], g) = \mathrm{Cov}(\mathbb{E}[f \mid \mathcal{F}_\mathcal{T}], \mathbb{E}[g \mid \mathcal{F}_\mathcal{T}]) \geq 0. \tag{8}$$

Proof. The equality is the tower property. We prove the inequality by induction on the maximum depth of the tree. It is immediate when the tree makes no query. Otherwise let X_e be the first queried coordinate and, after removing deterministic coordinates, write $q = \mathbb{P}(X_e = 1) \in (0,1)$. Put

$$\widehat{f} = \mathbb{E}[f \mid \mathcal{F}_{\mathcal{T}}], \qquad \widehat{g} = \mathbb{E}[g \mid \mathcal{F}_{\mathcal{T}}].$$

Conditional on $X_e = b$, the continuation is a decision tree on the remaining product coordinates, and the restrictions of f and g are increasing. The induction hypothesis therefore gives $\mathrm{Cov}(\widehat{f}, \widehat{g} \mid X_e = b) \geq 0$ for $b = 0, 1$. Moreover,

$$a_b := \mathbb{E}[\widehat{f} \mid X_e = b] = \mathbb{E}[f \mid X_e = b], \qquad d_b := \mathbb{E}[\widehat{g} \mid X_e = b] = \mathbb{E}[g \mid X_e = b].$$

The monotone coupling that uses the same remaining coordinates in the two branches gives $a_1 \geq a_0$ and $d_1 \geq d_0$. The law of total covariance now yields

$$\mathrm{Cov}(\widehat{f}, \widehat{g}) = \sum_{b=0}^{1} \mathbb{P}(X_e = b)\,\mathrm{Cov}(\widehat{f}, \widehat{g} \mid X_e = b) + q(1-q)(a_1 - a_0)(d_1 - d_0) \geq 0.$$

This completes the induction. □

Lemma 4.2 (Covariance after cluster exploration)**.** *Let U carry an independent hyperedge model. Let $x, v \in U$, let $S \subseteq U$ satisfy $x \in S$ and $v \notin S$, and let $f : 2^U \to [0,\infty)$ be increasing. For $W \subseteq U$, let*

$$H(W) = \mathrm{Cov}_W\big(f(C_x^W), \mathbf{1}\{v \leftrightarrow S\}\big),$$

with $H(W) = 0$ when $v \notin W$. Then $H(W) \geq 0$, and for every $B \subseteq W$,

$$Y_H^W(B) M_S^W(\varnothing) \leq M_S^W(B) H(W). \tag{9}$$

Proof. Harris' inequality gives $H(W) \geq 0$. We prove (9) for one fixed induced vertex set W; the case $M_S^W(\varnothing) = 0$ is immediate because the left side then vanishes and the right side is nonnegative. Suppose therefore that this probability is positive. Explore C_B^W, and let $\mathcal{F}_B$ be the information read by the exploration. Put

$$G = f(C_x^W), \qquad J = \mathbf{1}\{v \leftrightarrow S\}, \qquad Z = \mathbf{1}\{v \not\leftrightarrow S,\ v \leftrightarrow B\}, \qquad \widehat{G} = \mathbb{E}_W[G \mid \mathcal{F}_B].$$

Exact cluster conditioning and the law of total covariance give

$$H(W) = Y_H^W(B) + \mathrm{Cov}_W(\widehat{G}, J). \tag{10}$$

Indeed, on the branch $x \notin C_B^W$ the conditional covariance left after the exploration is $H(W \setminus C_B^W)$; on the complementary branch C_x^W has already been determined.

The event with indicator $J + Z$ is $\{v \leftrightarrow S \cup B\}$ and is increasing. Lemma 4.1, applied to the tree that explores C_B^W, yields

$$\mathrm{Cov}_W(\widehat{G}, J + Z) \geq 0.$$

The variable Z is $\mathcal{F}_B$-measurable, so $\mathrm{Cov}_W(\widehat{G}, Z) = \mathrm{Cov}_W(G, Z)$. Let $D = \{v \not\leftrightarrow S\}$, let $a = \mathbb{P}_W(D)$, and let $r = \mathbb{P}_W(Z)$. Conditional on D, the clusters of S and v are disjoint. Apply Lemma 3.5 to G and $1 - \mathbf{1}\{v \leftrightarrow B\}$. The first function increases with $\mathcal{K}_S$, whereas the second decreases with $\mathcal{K}_{\{v\}}$. The resulting positive association is equivalent to negative association between G and $\mathbf{1}\{v \leftrightarrow B\}$. Hence

$$\mathrm{Cov}_W(G, Z) \leq \frac{r}{a}\,\mathrm{Cov}_W(G, \mathbf{1}_D) = -\frac{r}{a} H(W).$$

Combining this inequality with (10) gives

$$Y_H^W(B) \le \left(1 - \frac{r}{a}\right) H(W).$$

Since $a - r = M_S^W(B)$, multiplication by $a = M_S^W(\varnothing)$ proves (9). $\square$

The stopped estimate is the only analytic input to the next induction. The rest is an application of the four-functions theorem on the set of exposed hyperedge labels.

Lemma 4.3 (Labeled two-source comparison)**.** *Let U carry an independent hyperedge model. Let $x, o, v \in U$, let $A \subseteq U$, and let $F : 2^U \to [0,\infty)$ satisfy $F(W) = 0$ when $v \notin W$. Suppose that, for every $W \subseteq U$ and every $B \subseteq W$,*

$$Y_F^W(B) M_A^W(\varnothing) \le M_A^W(B) F(W). \tag{11}$$

Let L be a finite label set with incidence map $\alpha : L \to 2^U$. Then, for all $N, N' \subseteq L$,

$$E_A^U(N) Y_F^U(N') \le M_A^U(N \cup N') X_F^U(N \cap N'). \tag{12}$$

Proof. We use strong induction on $|U|$, simultaneously for all finite label sets and incidence maps. If $v \in A \cup [N]_U$ or $o \in [N]_U$, then $E_A^U(N) = 0$. If $x \in [N']_U$, then $Y_F^U(N') = 0$. If $v \notin U$, then the assumption on F also makes $Y_F^U(N') = 0$. We may therefore remove these cases at every stage of the induction.

Suppose first that $N \cap N' = \varnothing$. Apply Lemma 3.4 to the avoided sets $A \cup [N]_U$ and $A \cup [N']_U$, using the constant function 1 and the indicator $\mathbf{1}\{o \in C_v\}$. Although the label sets are disjoint, their incidence sets may overlap. The intersection of the two avoided vertex sets is

$$A \cup ([N]_U \cap [N']_U),$$

which contains A. Monotonicity in the avoided set therefore gives

$$M_A^U(N') E_A^U(N) \le E_A^U(\varnothing) M_A^U(N \cup N'). \tag{13}$$

Multiply (11), with $B = [N']_U$, by $E_A^U(N)$ and use (13). If $M_A^U(\varnothing) > 0$, division by this factor and $X_F^U(\varnothing) = q_A(U)F(U)$ prove (12). If the factor vanishes, then $E_A^U(N) = 0$, so the conclusion is again immediate.

Now let $Z = N \cap N'$ be nonempty and let $K = [Z]_U$. Labels with empty incidence sets may be discarded, so K is nonempty. The zero cases already removed show that $x, o, v \notin K$. Expose all hyperedges incident to K, delete K, and write E_K for the exposed label set. In $U \setminus K$ use the new label set

$$(L \setminus Z) \,\dot\cup\, E_K.$$

An old label ℓ receives incidence set $\alpha(\ell) \setminus K$, while a fresh label $e \in E_K$ receives incidence set $\iota(e) \setminus K$. The avoided set in the smaller induced vertex set is $A \setminus K$; retaining A in the notation gives the same events.

For exposed states $I, J \subseteq E_K$, define

$$\widehat{N}_I = (N \setminus Z) \,\dot\cup\, I, \qquad \widehat{N}'_J = (N' \setminus Z) \,\dot\cup\, J.$$

The two kinds of labels form a disjoint union, so

$$\widehat{N}_I \cup \widehat{N}'_J = ((N \cup N') \setminus Z) \mathbin{\dot\cup} (I \cup J), \tag{14}$$
$$\widehat{N}_I \cap \widehat{N}'_J = I \cap J. \tag{15}$$

These are identities of labels, even when two fresh labels have the same incidence set.

Let $\pi(I)$ be the product probability of the exposed state I. Lemma 2.2 and exact cluster conditioning give

$$E_A^U(N) = \sum_{I \subseteq E_K} \pi(I)e(I), \qquad Y_F^U(N') = \sum_{J \subseteq E_K} \pi(J)y(J),$$
$$M_A^U(N \cup N') = \sum_{I \subseteq E_K} \pi(I)m(I), \qquad X_F^U(N \cap N') = \sum_{I \subseteq E_K} \pi(I)\chi(I),$$

where the four functions are the corresponding quantities in $U \setminus K$ with the sources displayed in (14) and (15). The induction hypothesis gives, for every $I, J \subseteq E_K$,

$$e(I)y(J) \leq m(I \cup J)\chi(I \cap J).$$

Moreover,

$$\pi(I)\pi(J) = \pi(I \cap J)\pi(I \cup J).$$

The finite four-functions theorem, Theorem 3.3 now gives (12). Since K is nonempty, the induction has passed to a strictly smaller vertex set, which completes the proof. $\square$

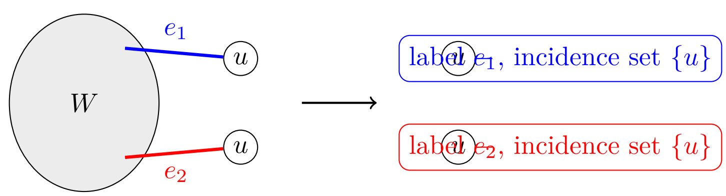

Figure 2: Two incident hyperedges can have the same outside trace. The vertex trace forgets which hyperedge was open. Separate labels preserve the intersection $I \cap J$ even when their incidence sets overlap.

5 Finite gluing

We now prove the finite inequality used in the lattice argument. The proof partitions a connection event according to the first member of an ordered set of intermediate vertices reached by the source cluster. This section also gives the conditioned-covariance estimate controlling the resulting selection bias.

Let $F : 2^V \to \mathbb{R}$ be increasing and nonnegative, and let A be a finite set of intermediate vertices. For $a \in A$, write

$$m_a = \mathbb{E}F(C_a).$$

Choose a total order $\prec$ on A such that $a \prec a_1$ implies $m_a \leq m_{a_1}$. Define

$$P_a^u = \{u \leftrightarrow a\} \cap \bigcap_{a_1 \prec a} \{u \not\leftrightarrow a_1\}. \tag{16}$$

These events are disjoint and their union is $\{u \leftrightarrow A\}$.

Theorem 5.1 (Ordered hitting decomposition). *For every finite independent hyperedge model, every vertex o, every finite set $A \subseteq V$ of intermediate vertices, every increasing nonnegative F, and every compatible order on A,*

$$\mathbb{E}[F(C_o); o \leftrightarrow A] \geq \sum_{a \in A} \mathbb{P}(P_a^o)\mathbb{E}F(C_a). \tag{17}$$

The conclusion includes hyperedge probabilities equal to zero or one.

We prove the theorem from two lemmas. Their formulation is included to make the dependence on Lemma 4.3 explicit.

For a finite set R of intermediate vertices, define the auxiliary difference at u by

$$\Delta_u(R) = \mathbb{E}[F(C_u); u \leftrightarrow R] - \sum_{a \in R} \mathbb{P}(P_a^u)m_a. \tag{18}$$

Lemma 5.2 (Transfer inequality for the auxiliary difference). *Let $R \subseteq V$, and let $o, v \in V$ with $v \notin R$. Then*

$$\mathbb{P}(v \not\leftrightarrow R)\Delta_o(R) \geq \mathbb{P}(o \leftrightarrow v, v \not\leftrightarrow R)\Delta_v(R). \tag{19}$$

Lemma 5.3 (Iterated conditioned-covariance inequality). *Let all hyperedge probabilities lie strictly between zero and one. Let $x, o, v, z_1, \ldots, z_\ell$ be distinct vertices, and let $Y \subseteq V$ be disjoint from them. Conditional on $x \not\leftrightarrow Y$, define*

$$\kappa_f(u) = \operatorname{Cov}\big(f(C_x), \mathbf{1}\{u \in C_x\} \mid x \not\leftrightarrow Y\big)$$

for an increasing function f. For $1 \leq j \leq \ell$, let

$$Y_j = Y \cup \{x, z_1, \ldots, z_{j-1}\}, \qquad c_j(u) = \mathbb{P}(u \in C_{z_j} \mid z_j \not\leftrightarrow Y_j).$$

Define $L^0 h = h$ and

$$(L^j h)(u) = (L^{j-1}h)(u) - c_j(u)(L^{j-1}h)(z_j). \tag{20}$$

Finally, let

$$\tau = \mathbb{P}(o \in C_v \mid v \not\leftrightarrow Y \cup \{x, z_1, \ldots, z_\ell\}).$$

Then

$$(L^\ell \kappa_f)(o) \geq \tau(L^\ell \kappa_f)(v). \tag{21}$$

Proof. We may add a constant to f, so we assume that f is nonnegative. We prove the iterated estimate by strong induction on ℓ. The proof has three components: conditioning on the remaining vertex set, expanding the recursively defined linear form, and comparing the coefficients at o and v.

Step 1: conditioning on the remaining vertex set. Let $\nu = \mathbb{P}(\,\cdot \mid x \not\leftrightarrow Y)$, and under ν expose C_Y. The remaining vertex set

$$W = V \setminus C_Y$$

contains x. Conditional on W, the hyperedges contained in W form the original independent model induced on W. Write π for the law of W under ν, and put

$$\kappa_f^W(u) = \operatorname{Cov}_W\big(f(C_x^W), \mathbf{1}\{u \in C_x^W\}\big).$$

The constants c_j and τ in the linear form remain the constants from the statement. Define the averaged difference

$$\mathcal{W}_\ell(f) = \mathbb{E}_\pi \left[(L^\ell \kappa_f^W)(o) - \tau (L^\ell \kappa_f^W)(v) \right]. \tag{22}$$

We record why nonnegativity of (22) implies the desired inequality. The operators L^j and the final linear combination at o and v determine constants λ_u, indexed by $u \in \{o, v, z_1, \dots, z_\ell\}$, such that

$$\mathfrak{M}_\ell(f) = \mathrm{Cov}_\nu \left(f(C_x), \sum_u \lambda_u \mathbf{1}\{u \in C_x\} \right).$$

Under ν, write $A = C_x$ and $B = C_Y$. Let P_A and P_B denote conditional expectation given A and B, respectively. The alternating conditional-expectation operator on functions of A is

$$\mathcal{A} = P_A P_B, \qquad (\mathcal{A} f)(A) = \mathbb{E}_\nu[\mathbb{E}_\nu[f(A) \mid B] \mid A].$$

The test function in the covariance is measurable with respect to A. Conditioning first on B therefore gives the exact identity

$$\mathfrak{M}_\ell(f) = \mathcal{W}_\ell(f) + \mathfrak{M}_\ell(\mathcal{A} f), \tag{23}$$

because the difference of the two covariances is

$$\mathbb{E}_\nu \left[\mathrm{Cov}_\nu \left(f(A), \sum_u \lambda_u \mathbf{1}\{u \in A\} \,\middle|\, B \right) \right] = \mathcal{W}_\ell(f).$$

The coupling from the proof of Lemma 3.5 shows that $B \mapsto \mathbb{E}_\nu[f(A) \mid B]$ is decreasing when f is increasing. During the next half-step, a larger A deletes more hyperedges and produces a smaller B. Thus $\mathcal{A}$ preserves increasing functions. It also preserves nonnegativity and constants. Because each hyperedge probability is less than one, the B-resampling closes every hyperedge incident to Y with probability at least

$$\epsilon_Y = \prod_{e:\, \iota(e) \cap Y \neq \varnothing} (1 - p_e) > 0.$$

This event forces $B = Y$. The following A-resampling then has a fixed law, independent of the old state. Thus the Markov kernel represented by $\mathcal{A}$ has a common component of mass at least ϵ_Y, and

$$\mathrm{osc}(\mathcal{A}^n f) \le (1 - \epsilon_Y)^n \, \mathrm{osc}(f).$$

The marginal law of $A = C_x$ under ν is stationary for this two-block kernel, so

$$\mathbb{E}_\nu[(\mathcal{A}^n f)(A)] = \mathbb{E}_\nu[f(A)] \qquad (n \ge 0).$$

Oscillation convergence together with this fixed mean shows that $\mathcal{A}^n f$ converges uniformly to the constant $\mathbb{E}_\nu[f(A)]$. When Y is empty, the first application of $\mathcal{A}$ is already constant, and the same conclusion holds. Iterating (23) now gives

$$\mathfrak{M}_\ell(f) = \sum_{i=0}^{n-1} \mathcal{W}_\ell(\mathcal{A}^i f) + \mathfrak{M}_\ell(\mathcal{A}^n f).$$

The last term tends to zero. Hence it is enough to prove $\mathcal{W}_\ell(g) \ge 0$ for every increasing nonnegative g.

Step 2: exact expansion. Fix such a function g and put

$$S_j = \{x, z_1, \dots, z_j\} \quad (0 \le j \le \ell), \qquad X = S_\ell.$$

For $u \in V$ define

$$\overline{h}_j(u) = \mathbb{E}_\pi \left[\mathrm{Cov}_W\big(g(C_x^W), \mathbf{1}\{u \leftrightarrow S_j\}\big) \right]. \tag{24}$$

Thus $\overline{h}_0(u) = \mathbb{E}_\pi[\kappa_g^W(u)]$. At the final level, if

$$h_u(W) = \mathrm{Cov}_W\big(g(C_x^W), \mathbf{1}\{u \leftrightarrow X\}\big),$$

then $\overline{h}_\ell(u) = \mathbb{E}_\pi[h_u(W)]$.

Let C_x^{-K} denote the cluster of x after all hyperedges meeting a vertex set K have been deleted, and define C_Y^{-K} in the same way. We will use

$$\Phi(K) = \mathbb{E}\left[\mathbf{1}\{x \not\leftrightarrow Y \text{ after deleting } K\} \left(g(C_x^{-C_Y^{-K}}) - g(C_x^{-K}) \right) \right]. \tag{25}$$

In the first cluster in (25), the hyperedges meeting C_Y^{-K} are deleted from the original configuration; hyperedges meeting K remain available unless they also meet that cluster.

For $K \subseteq K_1$, deleting K_1 shrinks $C_Y^{-K_1}$ and $C_x^{-K_1}$. On the avoidance event, C_x^{-K} is contained in $C_x^{-C_Y^{-K}}$. Consequently the first cluster in (25) can only grow, the second can only shrink, and the avoidance event can only grow when K is enlarged. The integrand is therefore pointwise nonnegative and increasing in K. Hence Φ is an increasing nonnegative function.

We next prove the one-level identity from which the expansion follows. For a set $Q \subseteq V$ containing x, abbreviate

$$m_Q = \mathbb{E}_Q g(C_x^Q).$$

Fix j, write $z = z_j$ and $S = S_{j-1}$, and let $D_j = \{z \not\leftrightarrow Y \cup S\} = \{z_j \not\leftrightarrow Y_j\}$. Exact conditioning on the cluster of z in the induced model on W gives

$$\begin{aligned} &\mathrm{Cov}_W\big(g(C_x^W), \mathbf{1}\{z \leftrightarrow S\}\big) \\ &\quad = \sum_{K \subseteq W:\ z \in K,\ K \cap S = \varnothing} \mathbb{P}_W(C_z^W = K)\big(m_W - m_{W \setminus K}\big), \end{aligned} \tag{26}$$

$$\begin{aligned} &\mathrm{Cov}_W\big(g(C_x^W), \mathbf{1}\{u \in C_z^W,\ C_z^W \cap S = \varnothing\}\big) \\ &\quad = \sum_{K \subseteq W:\ z, u \in K,\ K \cap S = \varnothing} \mathbb{P}_W(C_z^W = K)\big(m_{W \setminus K} - m_W\big). \end{aligned} \tag{27}$$

Both formulas are read as zero when $z \notin W$. Indeed, conditional on $C_z^W = K$, all hyperedges disjoint from K retain their product law, so the conditional mean of $g(C_x^W)$ is $m_{W \setminus K}$.

We use the following elementary exchange identity. For disjoint possible cluster values B and K,

$$\mathbb{P}(C_Y = B)\mathbb{P}_{V \setminus B}(C_z = K) = \mathbb{P}(C_z = K)\mathbb{P}_{V \setminus K}(C_Y = B). \tag{28}$$

Both sides are the joint probability that the two disjoint clusters have values B and K: expose either cluster first and use exact cluster conditioning for the other.

For every K disjoint from $Y \cup \{x\}$, conditioning (25) on $C_Y^{-K} = B$ gives the useful finite sum

$$\Phi(K) = \sum_{B \subseteq V \setminus K:\ x \notin B} \mathbb{P}_{V \setminus K}(C_Y = B) \left(m_{V \setminus B} - m_{V \setminus (B \cup K)} \right). \tag{29}$$

To justify the first term, observe that the coordinates meeting K were not read when C_Y^{-K} was formed. After the hyperedges meeting B are deleted, all remaining coordinates have exactly the product law on $V \setminus B$. The second term has the product law on $V \setminus (B \cup K)$.

Average (26) and (27) over $W = V \setminus C_Y$ under $\{x \not\leftrightarrow Y\}$. Equations (28) and (29) give

$$\mathbb{P}(x \not\leftrightarrow Y)\, \overline{h}_{j-1}(z_j) = \mathbb{E}\big[\Phi(C_{z_j}); D_j\big], \tag{30}$$

$$\mathbb{P}(x \not\leftrightarrow Y)\, \mathbb{E}_\pi\Big[\mathrm{Cov}_W\big(g(C_x^W), \mathbf{1}\{u \in C_{z_j}^W,\ C_{z_j}^W \cap S_{j-1} = \varnothing\}\big)\Big] = -\mathbb{E}\big[\Phi(C_{z_j})\mathbf{1}\{u \in C_{z_j}\}; D_j\big]. \tag{31}$$

These are identities of finite sums; in particular, no correlation inequality is being used here.

Let T_j be the linear operator

$$(T_j h)(u) = h(u) - c_j(u) h(z_j),$$

so that $L^j = T_j \cdots T_1$. The disjoint decomposition

$$\{u \leftrightarrow S_j\} = \{u \leftrightarrow S_{j-1}\} \,\dot\cup\, \{u \in C_{z_j},\ C_{z_j} \cap S_{j-1} = \varnothing\}$$

inside every induced model on W, together with (30)–(31), yields

$$(T_j \overline{h}_{j-1})(u) - \overline{h}_j(u) = \frac{\mathbb{P}(D_j)}{\mathbb{P}(x \not\leftrightarrow Y)}\, \mathrm{Cov}_{\rho_j}\big(\Phi(C_{z_j}), \mathbf{1}\{u \in C_{z_j}\}\big). \tag{32}$$

Indeed, after multiplying by $\mathbb{P}(x \not\leftrightarrow Y)$, the left side becomes

$$\mathbb{E}[\Phi(C_{z_j})\mathbf{1}\{u \in C_{z_j}\}; D_j] - c_j(u)\mathbb{E}[\Phi(C_{z_j}); D_j],$$

which is the right side of (32) multiplied by the same factor.

For $1 \le j \le \ell$, let $L_{>j} = T_\ell \cdots T_{j+1}$, with the identity operator when $j = \ell$, and put

$$\kappa_{\Phi,j}(u) = \mathrm{Cov}_{\rho_j}\big(\Phi(C_{z_j}), \mathbf{1}\{u \in C_{z_j}\}\big).$$

The elementary telescoping identity

$$T_\ell \cdots T_1 \overline{h}_0 - \overline{h}_\ell = \sum_{j=1}^{\ell} T_\ell \cdots T_{j+1}\big(T_j \overline{h}_{j-1} - \overline{h}_j\big) \tag{33}$$

now gives, after evaluation at o minus τ times evaluation at v,

$$\mathcal{W}_\ell(g) = \mathcal{H}_\ell(g) + \sum_{j=1}^{\ell} \frac{\mathbb{P}(z_j \not\leftrightarrow Y_j)}{\mathbb{P}(x \not\leftrightarrow Y)} \mathfrak{M}_{>j}^{(z_j, Y_j)}(\Phi), \tag{34}$$

where

$$\mathcal{H}_\ell(g) = \mathbb{E}_\pi\big[h_o(W) - \tau h_v(W)\big]$$

and

$$\mathfrak{M}_{>j}^{(z_j, Y_j)}(\Phi) = (L_{>j}\kappa_{\Phi,j})(o) - \tau(L_{>j}\kappa_{\Phi,j})(v).$$

Each difference in the sum in (34) has $\ell - j < \ell$ auxiliary vertices. Its distinguished source is z_j, its avoided set is Y_j, and its final avoided set is

$$Y_j \cup \{z_j, z_{j+1}, \ldots, z_\ell\} = Y \cup X.$$

Its coefficient at v is therefore the same τ. The induction hypothesis and the monotonicity of Φ show that every summand is nonnegative.

Step 3: comparison of the remaining coefficients. It remains to prove $\mathcal{H}_\ell(g) \geq 0$. For fixed W, let

$$q(W) = \mathbb{P}_W(o \leftrightarrow v \mid v \not\leftrightarrow X).$$

We first show

$$h_o(W) \geq q(W) h_v(W). \tag{35}$$

Write $m = \mathbb{E}_W g(C_x^W)$, let $D = \{v \not\leftrightarrow X\}$, and let $J = \{o \leftrightarrow v, v \not\leftrightarrow X\}$. Since

$$\{o \leftrightarrow X \cup \{v\}\} = \{o \leftrightarrow X\} \,\dot\cup\, J,$$

Harris' inequality gives

$$h_o(W) \geq -\mathbb{E}_W[(g(C_x^W) - m); J].$$

Conditional on D, apply Lemma 3.5 to $g(C_x)$ and $1 - \mathbf{1}\{o \in C_v\}$. The first function increases with $\mathcal{K}_X$, and the second decreases with $\mathcal{K}_{\{v\}}$. Equivalently, $g(C_x)$ and $\mathbf{1}\{o \in C_v\}$ are negatively associated. Hence

$$\mathbb{P}_W(D)\mathbb{E}_W[(g(C_x^W) - m); J] \leq \mathbb{P}_W(J)\mathbb{E}_W[(g(C_x^W) - m); D].$$

Now $h_v(W) = -\mathbb{E}_W[(g(C_x^W) - m); D]$ and $q(W) = \mathbb{P}_W(J)/\mathbb{P}_W(D)$, proving (35).

The conditional quantity $q(W)$ still has to be compared with the constant τ. Apply Lemma 4.3 with $A = X$ and $F(W) = h_v(W)$. Give each vertex of Y its own singleton label, and take the same label set for N and N'. Lemma 4.2 supplies (11); Harris gives $F \geq 0$, and $F(W) = 0$ when $v \notin W$. The diagonal conclusion is

$$\mathbb{E}[q(W)h_v(W); x \not\leftrightarrow Y] \geq \tau \mathbb{E}[h_v(W); x \not\leftrightarrow Y]. \tag{36}$$

After division by $\mathbb{P}(x \not\leftrightarrow Y)$, equations (35) and (36) give $\mathcal{H}_\ell(g) \geq 0$.

We have proved that $\mathcal{H}_\ell(g)$ and every other term in (34) are nonnegative. Thus $\mathcal{W}_\ell(g) \geq 0$. Step 1 converts this averaged statement to $\mathfrak{M}_\ell(f) \geq 0$, completing the strong induction and proving (21). □

Proof of Lemma 5.2. First suppose that all probabilities lie strictly between zero and one. We prove a stronger assertion that retains an ordered list of auxiliary vertices. In this strengthened assertion the distinguished vertices o, v, every member of the ordered list $Z = (z_1, \ldots, z_\ell)$, and the intermediate set R are pairwise disjoint except that no disjointness is required inside R itself. Build the operators L^j as in (20), but condition z_j to avoid

$$R \cup \{z_1, \ldots, z_{j-1}\}.$$

Use the coefficient

$$\mathbb{P}(o \leftrightarrow v \mid v \not\leftrightarrow R \cup \{z_1, \ldots, z_\ell\}).$$

Let $\mathcal{S}_{R,Z}(h)$ denote the resulting level form at o minus this coefficient times the level form at v. We claim

$$\mathcal{S}_{R,Z}(u \mapsto \Delta_u(R)) \geq 0 \tag{37}$$

for every auxiliary list satisfying the distinctness conditions of Lemma 5.3. We prove the claim by induction on $|R|$, simultaneously for all such lists. The empty intermediate set gives equality.

For the induction step, let k be the last intermediate vertex and let $T = R \setminus \{k\}$. Write

$$D_k = \{k \not\leftrightarrow T\}, \qquad c_k(u) = \mathbb{P}(u \leftrightarrow k \mid D_k),$$

and define

$$\gamma_k = \operatorname{Cov}\bigl(F(C_k), \mathbf{1}\{k \leftrightarrow T\}\bigr).$$

Splitting according to the first intermediate vertex reached gives the exact identity

$$\Delta_u(R) = \Delta_u(T) + \mathbb{P}(D_k) \operatorname{Cov}\bigl(F(C_k), \mathbf{1}\{u \in C_k\} \mid D_k\bigr) - \gamma_k c_k(u). \tag{38}$$

Apply $\mathcal{S}_{R,Z}$ to (38). Lemma 5.3 with distinguished source k, avoided set T, auxiliary list Z, and the cluster functional induced by F shows that the margin of the conditional-covariance term is nonnegative. The same lemma, applied to $\mathbf{1}\{C_k \neq \{k\}\}$, shows that

$$\mathcal{S}_{R,Z}(c_k) \geq 0. \tag{39}$$

Indeed, its conditional covariance profile is a positive constant times c_k; the constant is the conditional probability that every hyperedge incident to k is closed.

Compatibility of the order on the intermediate vertices gives

$$\gamma_k \leq \Delta_k(T), \tag{40}$$

because the difference is

$$\sum_{a \in T} \mathbb{P}(P_a^k)(m_k - m_a) \geq 0.$$

The definitions of the operators L^j give the algebraic identity

$$\mathcal{S}_{R,Z}\bigl(\Delta(T)\bigr) - \Delta_k(T)\mathcal{S}_{R,Z}(c_k) = \mathcal{S}_{T,(k,Z)}\bigl(\Delta(T)\bigr). \tag{41}$$

The right side is nonnegative by the induction hypothesis. Equations (38)–(41) therefore give

$$\mathcal{S}_{R,Z}\bigl(\Delta(R)\bigr) \geq \mathcal{S}_{T,(k,Z)}\bigl(\Delta(T)\bigr) \geq 0.$$

This proves (37).

Take Z to be empty. If o and v are distinct and lie outside R, then multiplying (37) by $\mathbb{P}(v \not\leftrightarrow R) > 0$ gives (19). If $o = v$, the two sides are equal. If $o \in R$, the event on the right of (19) is empty, and $\Delta_o(R) \geq 0$: in each event P_a^o, the selected intermediate vertex a has $m_a \leq m_o$, while $C_a = C_o$. Thus the transfer inequality holds for all positions allowed in its statement when the probabilities lie in the open cube.

For probabilities on the boundary of $[0,1]^E$, use the rank-free formula

$$\Delta_u(R) = \mathbb{E}\left[\left(F(C_u) - \min_{a \in C_u \cap R} m_a\right)\mathbf{1}\{u \leftrightarrow R\}\right]. \tag{42}$$

Approximate the weight vector by vectors in $(0,1)^E$. There are finitely many orders on R, so a subsequence has a fixed compatible order. Every term in the denominator-free form of (19) is a polynomial in the hyperedge probabilities. Continuity proves the boundary cases. $\square$

Proof of Theorem 5.1. We prove by induction on $|A|$ that $\Delta_o(A) \geq 0$. The empty case is immediate. Let k be the last intermediate vertex, and let $T = A \setminus \{k\}$. Put

$$D = \{k \not\leftrightarrow T\}, \qquad H = \{o \leftrightarrow k, k \not\leftrightarrow T\}.$$

Splitting off the event P_k^o gives

$$\Delta_o(A) = \Delta_o(T) - \mathbb{E}[(m_k - F(C_k)); H]. \tag{43}$$

Conditional positive association from Lemma 3.4 yields

$$\mathbb{P}(D)\mathbb{E}[(m_k - F(C_k)); H] \leq \mathbb{P}(H)\gamma, \tag{44}$$

where

$$\gamma = m_k\mathbb{P}(D) - \mathbb{E}[F(C_k); D].$$

The ordered partition and compatibility give $\gamma \leq \Delta_k(T)$. Lemma 5.2 gives

$$\mathbb{P}(D)\Delta_o(T) \geq \mathbb{P}(H)\Delta_k(T).$$

Combining this inequality with (43)–(44) proves $\mathbb{P}(D)\Delta_o(A) \geq 0$. If $\mathbb{P}(D) > 0$, the result follows by division. If $\mathbb{P}(D) = 0$, then $\mathbb{P}(H) = 0$ and the induction hypothesis gives the result. This completes the induction and proves (17). $\square$

Theorem 5.4 (Hyperedge gluing)**.** *Let $o, b \in V$, and let $A \subseteq V$ be nonempty. In every finite independent hyperedge model,*

$$\mathbb{P}(o \leftrightarrow b) \geq \mathbb{P}(o \leftrightarrow A) \min_{a\in A} \mathbb{P}(a \leftrightarrow b). \tag{45}$$

Proof. Apply Theorem 5.1 to $F(K) = \mathbf{1}\{b \in K\}$. The events P_a^o partition $\{o \leftrightarrow A\}$, so

$$\mathbb{P}(o \leftrightarrow b, o \leftrightarrow A) \geq \sum_{a\in A} \mathbb{P}(P_a^o)\mathbb{P}(a \leftrightarrow b) \geq \mathbb{P}(o \leftrightarrow A) \min_{a\in A} \mathbb{P}(a \leftrightarrow b).$$

Dropping $\{o \leftrightarrow A\}$ from the event on the left proves (45). $\square$

6 Site gluing with conditioned-open terminals

We now return to ordinary site percolation on a finite graph. Vertices may have unequal opening probabilities.

Corollary 6.1 (Site gluing with conditioned-open terminals)**.** *Let G be a finite graph with independent site probabilities. Let $o, b \in V(G)$, and let $A \subseteq V(G)$ be nonempty. Suppose that o, b, and every vertex of A are open with probability one. Then*

$$\mathbb{P}(o \leftrightarrow b) \geq \mathbb{P}(o \leftrightarrow A) \min_{a\in A} \mathbb{P}(a \leftrightarrow b). \tag{46}$$

Consequently, if both factors on the right exceed $1 - \delta$, then $\mathbb{P}(o \leftrightarrow b) > (1 - \delta)^2$.

Proof. Use the hyperedge representation in Lemma 2.1. All connectivity events in (46) agree with their hyperedge counterparts, including possible coincident terminals because the named vertices are forced open. Theorem 5.4 proves the first claim. The second follows by multiplication. $\square$

Two wiring operations will be used below. To wire a source set S, add a new vertex o that is open with probability one and adjacent to each vertex of S. Then $o \leftrightarrow T$ is exactly the event that S connects to T. To wire a target set T, add a new vertex b that is open with probability one and adjacent to each vertex of T. Then $S \leftrightarrow b$ is exactly the event that S connects to T. Wiring T can increase the probability that the source reaches the intermediate set; this only strengthens the hypothesis in Corollary 6.1.

Conditioning on finitely many site states gives another operation. Delete the closed conditioned vertices and contract each connected component of open conditioned vertices to one vertex that is open with probability one. The unconditioned vertices remain independent with their original probabilities. Connections among the remaining vertices are unchanged. Thus the source, target, and intermediate vertices in the conditional graph satisfy the pinning hypothesis of Corollary 6.1.

7 Elementary infinite-volume facts

This section proves the only infinite-volume facts used later. In particular, the argument does not import a critical half-space theorem or a two-dimensional critical theorem.

7.1 Translation zero–one law and finite modifications

Let $\mathbb{P}_p^{(d)}$ denote Bernoulli site percolation on $\mathbb{Z}^d$.

Lemma 7.1 (Translation zero–one law)**.** *Every event invariant under all translations of $\mathbb{Z}^d$ has $\mathbb{P}_p^{(d)}$-probability zero or one.*

Proof. Let A be translation invariant and let $\varepsilon > 0$. Cylinder events are dense in the product sigma-field, so there is an event B, depending on a finite set F, such that

$$\mathbb{P}_p^{(d)}(A \triangle B) < \varepsilon.$$

Choose a translation τ with $F \cap \tau F = \varnothing$. The events B and τB are independent, while $\tau A = A$. Moreover

$$(B \cap \tau B) \triangle A \subseteq (B \triangle A) \cup (\tau B \triangle A).$$

Hence

$$\left|\mathbb{P}(A) - \mathbb{P}(B)^2\right| \le 2\varepsilon, \qquad \left|\mathbb{P}(B)^2 - \mathbb{P}(A)^2\right| \le 2\varepsilon.$$

Thus $|\mathbb{P}(A) - \mathbb{P}(A)^2| \le 4\varepsilon$. Letting $\varepsilon \downarrow 0$ proves the claim. □

Lemma 7.2 (Finite-energy modification)**.** *Let $0 < p < 1$, let $F \subset \mathbb{Z}^d$ be finite, and let $\zeta \in \{0,1\}^F$. If an event A has positive probability, then the set of configurations obtained from members of A by resetting the coordinates of F to ζ also has positive probability.*

Proof. For each pattern $\eta \in \{0,1\}^F$, let

$$A_\eta = \{\omega_{F^c} : (\eta, \omega_{F^c}) \in A\}.$$

Finite product decomposition gives

$$\mathbb{P}_p^{(d)}(A) = \sum_{\eta \in \{0,1\}^F} \mathbb{P}_p^{(d)}(\omega_F = \eta)\mathbb{P}_p^{F^c}(A_\eta).$$

Some η therefore has $\mathbb{P}_p^{F^c}(A_\eta) > 0$. The event $\{\omega_{F^c} \in A_\eta,\ \omega_F = \zeta\}$ has probability

$$\mathbb{P}_p^{F^c}(A_\eta) \prod_{x \in F} p^{\zeta_x}(1-p)^{1-\zeta_x} > 0,$$

and every one of its configurations is obtained by resetting the F-coordinates of a configuration in A to ζ. $\square$

7.2 Uniqueness of the infinite cluster

Call a vertex x a *trifurcation* when it is open and deleting x from its open cluster leaves at least three infinite components.

Lemma 7.3 (Tree boundary count)**.** *For every site configuration and every $n \geq 1$, the number of trifurcations in Λ_n is at most $|\partial_{\mathrm{in}}\Lambda_{n+1}|$.*

Proof. Consider one connected component G of the open graph induced by Λ_{n+1}, and put

$$B_G = G \cap \partial_{\mathrm{in}}\Lambda_{n+1}.$$

Choose an inclusion-minimal connected subgraph $T_G \subseteq G$ containing B_G. It is a tree, and every leaf of T_G belongs to B_G.

If $x \in G \cap \Lambda_n$ is a trifurcation, three infinite components of its full open cluster after deletion of x each contain a path to $\partial_{\mathrm{in}}\Lambda_{n+1}$. Their initial segments lie in three distinct components of $G \setminus \{x\}$ and end in B_G. Consequently every connected subgraph containing B_G must contain x and must use an edge from x into each of those three components. Hence $\deg_{T_G}(x) \geq 3$.

For a finite tree, the number of vertices of degree at least three is at most its number of leaves. Therefore the number of trifurcations in $G \cap \Lambda_n$ is at most $|B_G|$. Summing over the disjoint open components G proves the result. $\square$

Lemma 7.4 (Finite-energy construction of a trifurcation)**.** *Let $d \geq 2$ and $0 < p < 1$. If a finite set meets three distinct infinite open clusters with positive probability, then some fixed vertex is a trifurcation with positive probability.*

Proof. For $N \geq 0$, write

$$D_N = \{x \in \mathbb{Z}^d : \|x\|_1 \leq N\}.$$

On the event in the hypothesis, choose one one-sided infinite self-avoiding open path in each of three distinct clusters. Along each path there is a last vertex in D_N; when N is large enough this vertex belongs to $\partial_{\mathrm{in}}D_N$, and the remaining tail lies outside D_N. Taking a countable union over N and then a finite union over boundary triples, we obtain an N and three fixed, distinct sites $x_1, x_2, x_3 \in \partial_{\mathrm{in}}D_N$ such that, on an event A of positive probability,

(T1) the x_i lie in three distinct infinite open clusters;

(T2) for each i there is an infinite open self-avoiding path starting at x_i whose later vertices lie outside D_N.

Increase N before the finite-union selection if necessary so that $N \geq 2$. Each x_i has a neighbor in D_{N-1}, and D_{N-1} is connected. Hence there is a connected vertex set contained in D_N that contains the three x_i and meets $\partial_{\mathrm{in}}D_N$ only at those sites. Among all such sets choose one, say T, of minimum cardinality.

The graph induced by T is a tree. To see this, suppose that it contains a cycle C. For $u \in C$ that is not a terminal, minimality says that $T \setminus \{u\}$ does not connect all three terminals. Since $C \setminus \{u\}$ is still connected, some component of the induced graph on $T \setminus C$, attached to C only at u, must contain a terminal; assign one such terminal to u. Assign to a terminal lying on C the terminal itself. The assignments are injective: components attached at distinct cycle vertices are disjoint, and none contains a terminal lying on C. Thus the vertices of C inject into $\{x_1, x_2, x_3\}$, so $|C| \leq 3$. This is impossible because the nearest-neighbor lattice is bipartite and every cycle has even length at least four. Hence no cycle exists.

In this tree, let v be the median of the three terminals, namely the unique common vertex of the three pairwise paths. If v is not a terminal, then $T \setminus \{v\}$ has three components containing the x_i. If $v = x_i$ for some i, then $T \setminus \{v\}$ has two components containing the other two terminals, while the infinite tail from (T2) beginning after x_i supplies a third component after v is deleted.

Reset every site of T to open and every site of $D_N \setminus T$ to closed. The only open sites on $\partial_{\text{in}} D_N$ are now the three x_i. Any open path joining two components of $T \setminus \{v\}$ without using v would therefore have to leave D_N through one x_i and return through another. Outside D_N the configuration was not changed, so such a path would join two of the three original clusters, contrary to (T1). Every component identified in the preceding paragraph contains one of the unchanged infinite tails. Hence v is a trifurcation after the reset.

The reset concerns finitely many sites. Lemma 7.2 therefore makes this trifurcation event positive. The vertex v is fixed, because N, the terminal triple, and a deterministic minimizing rule for T were fixed before applying the modification lemma. □

Theorem 7.5 (Uniqueness of the infinite site cluster)**.** *Let $d \geq 2$ and $0 < p < 1$. Bernoulli site percolation on $\mathbb{Z}^d$ has almost surely at most one infinite open cluster.*

Proof. Let N_∞ be the number of infinite open clusters. It is translation invariant, so Lemma 7.1, applied to the events $\{N_\infty = k\}$ and $\{N_\infty = \infty\}$, shows that N_∞ is almost surely a constant in $\{0, 1, 2, \ldots, \infty\}$.

Suppose first that this constant is a finite $k \geq 2$. With probability tending to one as $n \to \infty$, every infinite cluster meets Λ_n. For some n this event has positive probability. Reset every site of Λ_n to open. All k old infinite clusters are then joined through the connected box Λ_n. A finite modification cannot create an additional infinite cluster disjoint from all old infinite clusters: after removing the modified finite set, an infinite component must contain an infinite component of the old configuration. Thus the modified configuration has exactly one infinite cluster. Lemma 7.2 gives this event positive probability, contradicting the almost-sure constancy of N_∞.

Suppose now that $N_\infty = \infty$ almost surely. Since the boxes exhaust the lattice, some finite box meets at least three infinite clusters with positive probability. Lemma 7.4 and translation invariance therefore give a number $a > 0$ such that every vertex is a trifurcation with probability a.

Lemma 7.3 now gives

$$a|\Lambda_n| = \mathbb{E}[\#\{\text{trifurcations in } \Lambda_n\}] \leq |\partial_{\text{in}} \Lambda_{n+1}|.$$

The right side divided by $|\Lambda_n|$ tends to zero, a contradiction. Thus $N_\infty \in \{0, 1\}$ almost surely. □

7.3 An explicit supercritical planar parameter

For a finite nearest-neighbor connected set $C \subset \mathbb{Z}^2$, define its external vertex boundary by

$$\partial_\infty C = \{y \notin C : y \sim C \text{ and } y \text{ is joined to infinity in } \mathbb{Z}^2 \setminus C\}.$$

Call two vertices star-adjacent when their sup-distance is one.

Lemma 7.6 (External boundary connectivity)**.** *If $C \subset \mathbb{Z}^2$ is finite, nearest-neighbor connected, and contains the origin, then $\partial_\infty C$ is nonempty and star-connected. Moreover it contains a point $k_+ e_1$ and a point $-k_- e_1$, where $k_+, k_- \geq 1$.*

Proof. Place the closed unit square centered at each vertex of C, and consider the unbounded component of the complement of their union. The lattice edges on its boundary form a finite closed edge-walk. To each boundary edge assign the unit lattice square on its unbounded side, equivalently the center of that square. This center lies outside C, is nearest-neighbor adjacent to C, and can be joined through the unbounded complementary component to arbitrarily distant lattice squares; hence it belongs to $\partial_\infty C$. Consecutive boundary edges have unbounded-side square centers that are equal or star-adjacent. Conversely, every member of $\partial_\infty C$ supplies exactly such a boundary edge: its unit square lies on the unbounded side of the edge it shares with C. Thus these centers are precisely $\partial_\infty C$. Traversing the outer boundary shows that this set is nonempty and star-connected.

Let

$$k_+ = 1 + \max\{j \geq 0 : je_1 \in C\}, \qquad k_- = 1 + \max\{j \geq 0 : -je_1 \in C\}.$$

The rays beyond $k_+ e_1$ and $-k_- e_1$ avoid C, and each displayed site is adjacent to the last point of C on the corresponding ray. They therefore belong to $\partial_\infty C$. □

Lemma 7.7 (Counting star-connected sets)**.** *For a fixed vertex $x \in \mathbb{Z}^2$, the number of star-connected sets of cardinality $m \geq 1$ that contain x is at most $8^{2(m-1)}$.*

Proof. Every such set has a spanning tree in the star graph. Starting at x, a depth-first traversal of the tree is a star-neighbor walk of length $2(m-1)$ visiting every vertex of the set. There are at most eight choices at each step. Every star-connected set occurs as the visited set of at least one such walk, proving the bound. □

Proposition 7.8 (A fixed high-density planar process percolates)**.** *Bernoulli site percolation on $\mathbb{Z}^2$ percolates at*

$$p_\star = \frac{999}{1000}.$$

Consequently it percolates at every parameter at least $p_\star$.

Proof. Let $C(0)$ be finite and suppose that the origin is open. Every site of $\partial_\infty C(0)$ is closed, since an open neighbor of $C(0)$ would belong to the origin cluster. Put $m = |\partial_\infty C(0)|$. By Lemma 7.6, this star-connected set contains $k_+ e_1$ and $-k_- e_1$. A star path between these two sites changes the first coordinate by at most one at each step, so $k_+ + k_- + 1 \leq m$ and in particular $1 \leq k_+ \leq m$.

For a fixed m, choose the root $k_+ e_1$ in at most m ways. By Lemma 7.7, there are at most $m 8^{2(m-1)} \leq m 64^m$ possible external boundaries of cardinality m. At parameter $p_\star$, each specified boundary is entirely closed with probability 1000^{-m}. Hence

$$\mathbb{P}^{(2)}_{p_\star}(0 \text{ open and } |C(0)| < \infty) \leq \sum_{m \geq 1} m \left(\frac{64}{1000}\right)^m = \frac{x}{(1-x)^2}\bigg|_{x=8/125} = \frac{1000}{13689} < \frac{1}{10}.$$

It follows that

$$\theta^{(2)}_{\text{site}}(p_\star) \geq p_\star - \frac{1}{10} > 0.$$

Monotonicity gives the final assertion. □

Lemma 7.9 (The critical parameter is interior)**.** *For every $d \geq 2$,*

$$0 < p_c^{\mathrm{site}}(\mathbb{Z}^d) < 1.$$

Proof. The number of self-avoiding nearest-neighbor paths of length n starting at the origin is at most $2d(2d-1)^{n-1}$. Hence, when $p(2d-1) < 1$, the probability that the origin has an open path of length n is at most

$$2d(2d-1)^{n-1}p^{n+1},$$

which tends to zero. Thus $p_c^{\mathrm{site}}(\mathbb{Z}^d) > 0$.

The coordinate copy of $\mathbb{Z}^2$ inside $\mathbb{Z}^d$ percolates at $p_\star$ by Proposition 7.8. Therefore $p_c^{\mathrm{site}}(\mathbb{Z}^d) \leq p_\star < 1$. $\square$

8 Block renormalization for site percolation

We now prove the infinite-volume implication used at criticality. The argument adapts the same-parameter block construction of Kozma and Nitzan [6, Sections 2–4] to site variables. We include the finite statements, the site-specific separation of the revealed variables, and the parameter order because the conclusion must hold at the same value of p.

For $m \geq 0$, put $\Lambda_m = [-m, m]^d \cap \mathbb{Z}^d$. If $B = [a, b] \cap \mathbb{Z}^d$ is an integer box, write

$$B^{+r} = [a - r\mathbf{1}, b + r\mathbf{1}] \cap \mathbb{Z}^d.$$

For a finite set $K \subseteq \mathbb{Z}^d$, its inner and outer vertex boundaries are

$$\partial_{\mathrm{in}} K = \{x \in K : \|x - y\|_1 = 1 \text{ for some } y \notin K\},$$
$$\partial_{\mathrm{out}} K = \{y \notin K : \|x - y\|_1 = 1 \text{ for some } x \in K\}.$$

For sets $S, T, D \subseteq \mathbb{Z}^d$, the notation $S \overset{D}{\leftrightarrow} T$ means that an open site path contained in D has one endpoint in each of S and T.

8.1 Finite block events

For this section, a **finite site block at parameter** p means a finite graph G, an integer box $D \subseteq V(G) \cap \mathbb{Z}^d$, and independent site states such that:

(S1) the sites of D have opening probability p;

(S2) the graph induced by D is the nearest-neighbor lattice graph;

(S3) an edge of G from $V(G) \setminus D$ to D ends at the inner vertex boundary of D.

The sites outside D may have different opening probabilities. Wiring and conditioning are allowed there. A source o in a finite site block is always outside D and is forced open.

A **geometry** $\gamma = (Q, F)$ is a pair of integer boxes. At least one coordinate of F lies entirely in $[1, \infty)$ or entirely in $(-\infty, -1]$. Define

$$Q_\gamma(\ell, v) = v + \ell Q, \qquad F_\gamma(\ell, v) = v + \ell F.$$

We say that γ satisfies the **uniform connection property at** p when, for each $\alpha > 0$, there are integers $k, \ell_0 \geq 1$ such that, for all integers $m \geq k$ and $\ell \geq \ell_0$,

$$\mathbb{P}_p(\Lambda_m \overset{Q_\gamma(\ell,0)}{\leftrightarrow} F_\gamma(\ell, 0)) > 1 - \alpha. \tag{47}$$

Let $B \subseteq D$ be an integer box, let $R \geq 1$ be an integer, and let $\mathscr{H}$ be a finite list of geometries. We will use a nonempty set $T \subseteq D$ only when it satisfies the following explicit covering condition: for each $v \in B^{+R}$, there are $\ell \geq R$ and $\gamma \in \mathscr{H}$ such that

$$Q_\gamma(\ell, v) \subseteq D, \qquad F_\gamma(\ell, v) \subseteq T. \tag{48}$$

These quantifiers are uniform in v.

We use two standard finite-size facts for supercritical site percolation. The first follows from uniqueness, and the second is the square-root argument for the orthants of a face; compare [5, Sections 7.2 and 8.1]. We state and prove the precise forms used below.

Lemma 8.1 (Local uniqueness and face-connection estimates). *Let $d \geq 2$ be an integer, and let $p \in (0, 1)$ satisfy $\theta_{\text{site}}(p) > 0$.*

(L1) *For $m \geq 1$ and $\alpha > 0$, there is $M > m$ such that, with probability greater than $1 - \alpha$, all open components in $v + \Lambda_M$ that meet both $v + \Lambda_m$ and the inner boundary of $v + \Lambda_M$ belong to one component of $v + \Lambda_M$. The estimate is uniform in v.*

(L2) *Let $i \in \{1, \ldots, d\}$ and $\epsilon \in \{-1, 1\}^d$. In the cube $Q = [-1, 1]^d$, set*

$$F_{i,\epsilon} = \{z \in Q : \epsilon_i z_i = 1,\ 0 \leq \epsilon_j z_j \leq 1\ (j \neq i)\}.$$

Each pair $(Q, F_{i,\epsilon})$ satisfies the uniform connection property at p.

Proof. The event that an infinite open cluster exists is translation invariant and has positive probability. Lemma 7.1 and Theorem 7.5 therefore show that there is exactly one infinite open cluster almost surely. Fix two vertices $a, b \in v + \Lambda_m$. If one of their full-lattice clusters is finite, then that cluster misses the boundary of $v + \Lambda_M$ for all sufficiently large M. If both clusters are infinite, then uniqueness joins a to b by a finite path, which lies in $v + \Lambda_M$ for all sufficiently large M. Thus the probability that a and b lie in two distinct components of $v + \Lambda_M$, both reaching its inner boundary, tends to zero. A union bound over the finitely many pairs $a, b \in v + \Lambda_m$ proves (L1). Translation invariance makes the choice of M uniform in v.

We prove (L2) by the square-root argument. By the preceding zero–one and uniqueness argument, the unique infinite cluster exists almost surely. The increasing boxes Λ_m therefore meet it with probability tending to one. Let $q = d\,2^d$ be the number of indexed orthant-faces in the statement. Choose m_0 so that

$$\mathbb{P}_p(\Lambda_{m_0} \not\leftrightarrow \infty) < \alpha^q.$$

For every $\ell \geq m_0$, an infinite path starting in Λ_{m_0} contains a path in Λ_ℓ from Λ_{m_0} to $\partial_{\text{in}}\Lambda_\ell$. Hence the probability that no such path reaches the full inner boundary is less than α^q.

For each indexed orthant-face, let $D_{i,\epsilon}$ be the decreasing event that it is not reached from Λ_{m_0} inside Λ_ℓ. The intersection of the q events $D_{i,\epsilon}$ is precisely the event that the full inner boundary is not reached. Harris' inequality for decreasing site events and lattice symmetry give

$$\mathbb{P}_p(D_{i,\epsilon})^q \leq \mathbb{P}_p\left(\bigcap_{i,\epsilon} D_{i,\epsilon}\right) < \alpha^q.$$

Thus every orthant-face is reached with probability greater than $1 - \alpha$. Enlarging the source box only enlarges the event. We may take $k = \ell_0 = m_0$ in (47). □

The following elementary lattice construction separates the sites in the finite sets J_x from those later fixed by conditioning. This separation is the site-percolation change that is absent from the bond proof.

Lemma 8.2 (Boundary connection estimates for thick shells)**.** *Let*

$$B = \prod_{h=1}^{d} [a_h, b_h]_{\mathbb{Z}}, \qquad M \geq 1, \qquad j \geq 2M + 2.$$

Put

$$D_j = B^{+j}, \qquad O_j = B^{+(j-1)}, \qquad I_j = B^{+(j-2M-2)}, \qquad S_j = O_j \setminus I_j.$$

For each $x \in \partial_{\text{out}} D_j$ there are a center v_x, an oriented subface U_x of $v_x + \Lambda_M$, and a finite set $J_x \subseteq D_j \setminus O_j$ such that:

(W1) *$v_x + \Lambda_M \subseteq S_j$ and U_x is a subface of $v_x + \Lambda_M$;*

(W2) *if x, every site of J_x, and $u \in U_x$ are open, then $x \overset{\{x\}\cup J_x \cup \{u\}}{\longleftrightarrow} u$;*

(W3) *$J_x \cap S_j = \varnothing$, and an exploration confined to $\mathbb{Z}^d \setminus D_j$ has not inspected J_x;*

(W4) *every site of J_x is at sup-distance at most $R_M = 2M + 2$ from x, and hence $|J_x| \leq s_d(M) := |\Lambda_{R_M}|$;*

(W5) *if $\|x - y\|_\infty > 2R_M$, then $J_x \cap J_y = \varnothing$;*

(W6) *the shell pattern $\omega|_{S_j}$ fixes the full configuration on $v_x + \Lambda_M$.*

Proof. There is a unique normal pair (i, σ), $\sigma \in \{-1, 1\}$, for which x lies one lattice step outside the (i, σ) face of D_j. For $h \neq i$, clamp x_h to

$$[a_h - j + M + 1,\ b_h + j - M - 1]_{\mathbb{Z}};$$

the interval is nonempty since $j \geq 2M + 2$. Call the clamped coordinate q_h, let q_i be the coordinate of the (i, σ) face of O_j, and put $v_x = q - \sigma M e_i$. The transverse margin puts the full cube in O_j. In the normal direction its innermost coordinate is one lattice step outside the corresponding face of I_j, so the cube is contained in S_j. Divide its outward face by the transverse coordinate hyperplanes and choose the closed subface U_x containing the face center q.

Let $\widehat{U}_x = U_x + \sigma e_i$. This cap lies in the single outer layer $D_j \setminus O_j$. Starting from $x - \sigma e_i$, change the transverse coordinates one at a time, on the same outer face of D_j, until reaching $q + \sigma e_i$. Let J_x be this path together with the cap, omitting x. The cap is a nearest-neighbor connected $(d-1)$-dimensional box containing $q + \sigma e_i$. It is disjoint from S_j, and every $u \in U_x$ is adjacent to $u + \sigma e_i \in \widehat{U}_x$. Clamping moves a transverse coordinate by at most $M + 1$ and the cap extends by at most a further M, proving the radius bound. The remaining assertions follow. $\square$

8.2 Extending a high-probability connection

Lemma 8.3 (Finite-volume connection extension)**.** *Let $d \geq 2$ be an integer, and let $p \in (0, 1)$ satisfy $\theta_{\text{site}}(p) > 0$. For each $\varepsilon > 0$, there is $\delta > 0$ with the following property. For every finite list $\mathscr{H}$ of geometries satisfying the uniform connection property at p, there is an integer $R \geq 1$ such that the next implication holds. Let (G, D) be a finite site block at parameter p, and let*

$o \in V(G) \setminus D$ be forced open. Let $B \subseteq D$ be a nonempty integer box satisfying $B^{+R} \subseteq D$, and let $T \subseteq D$ satisfy (48) *for $(B, D, R, \mathscr{H})$. Then*

$$\mathbb{P}(o \leftrightarrow B) > 1 - \delta \quad \Longrightarrow \quad \mathbb{P}(o \leftrightarrow T) > 1 - \varepsilon. \tag{49}$$

The number δ depends only on d, p, ε; in particular, it is chosen before $\mathscr{H}$.

Proof. It is enough to treat $0 < \varepsilon \le 1$. Put

$$\delta_c = \frac{\varepsilon}{8}, \qquad \delta = \frac{\varepsilon \delta_c}{12}. \tag{50}$$

Then $\delta \le \delta_c$, $3\delta \le 1$, and

$$(1 - \delta_c)^2 > 1 - \frac{\varepsilon}{2}, \qquad 12\delta = \varepsilon\delta_c. \tag{51}$$

Choose common hittability scales for the finite list $\mathscr{H}$ and for the finite subface list in Lemma 8.1, with error δ^2. Choose m beyond all source thresholds. Then choose $M > m$ beyond all subface scale thresholds so that (L1) also has error δ^2. Let $s = s_d(M)$. Choose k and then N so that

$$(1 - p^s)^k \le \delta, \qquad N \ge k|\Lambda_{2R_M}|. \tag{52}$$

Finally choose an integer $L \ge 1$ satisfying

$$L \ge \frac{1}{\delta(1-p)^{2dN}}. \tag{53}$$

Use the consecutive levels

$$j_t = 2M + 2 + t, \qquad 0 \le t < L.$$

Choose $R \ge 2M + L + 2$ larger than every scale threshold belonging to $\mathscr{H}$. Then $j_t + 1 \le R$, so each contact, set J_x, cube, and shell used at level j_t lies in $B^{+R} \subseteq D$. This is the only point at which R depends on $\mathscr{H}$.

For a level $j = j_t$, explore the open cluster of o in the complement of B^{+j}. Let K_j be the set of reached sites in $\partial_{\text{out}} B^{+j}$, and call them contacts. If $|K_j| < N$, at most $2dN$ sites in B^{+j} are adjacent to K_j. Those sites have not been read by the exterior exploration. Closing all of them has conditional probability at least

$$q_N = (1-p)^{2dN}$$

and blocks entry into B^{+j}.

Here is the stopping-time form of the counting argument behind the choice of j. Process the selected levels from outside to inside, extending the exterior cluster exploration as needed. Let X be the number of levels for which $|K_{j_t}| < N$, and let σ_r be the level of the rth such failure. Immediately after σ_r is recognized, let I_r be the set of sites inside $B^{+\sigma_r}$ adjacent to its contacts. The set I_r is determined by the current transcript, has cardinality at most $2dN$, and none of its sites has yet been queried. The sets encountered at successive failure levels are fresh, because the exploration moves strictly inward. Thus, conditional on the transcript just before I_r is read,

$$\mathbb{P}(I_r \text{ is not entirely closed} \mid \text{transcript}) \le 1 - q_N.$$

If I_r is entirely closed, no open path from o can enter $B^{+\sigma_r}$, so the event $\{o \leftrightarrow B\}$ forces I_r not to be entirely closed. Iterating the preceding conditional estimate at the stopping times $\sigma_1, \dots, \sigma_t$ gives

$$\mathbb{P}(o \leftrightarrow B, X \ge t) \le (1 - q_N)^t.$$

Indeed, the event on the left is contained in the event that each of the first t predictably selected fresh sets I_r contains an open site, and the tower property removes these requirements one at a time. The tail-sum formula now gives

$$\sum_{t=0}^{L-1} \mathbb{P}(o \leftrightarrow B, |K_{j_t}| < N) = \mathbb{E}[X; o \leftrightarrow B] = \sum_{t\geq 1} \mathbb{P}(o \leftrightarrow B, X \geq t) \leq \sum_{t\geq 1}(1-q_N)^t < q_N^{-1}.$$

By (53), one level j therefore satisfies

$$\mathbb{P}(o \leftrightarrow B,\, |K_j| < N) < \delta.$$

Since $\mathbb{P}(o \leftrightarrow B) > 1-\delta$, this level satisfies

$$\mathbb{P}(|K_j| \geq N) > 1 - 2\delta. \tag{54}$$

On $\{|K_j| \geq N\}$, the greedy packing bound and (52) select k contacts separated by more than $2R_M$. Apply Lemma 8.2. Their sets J_x are disjoint, are disjoint from S_j, and have not been read by the exterior exploration. Thus

$$\mathbb{P}(\text{none of the selected sets } J_x \text{ is entirely open} \mid K_j) \leq (1-p^s)^k \leq \delta. \tag{55}$$

Let $\mathcal{G}$ be the event that at least one selected contact has every site of its set J_x open, and use a fixed enumeration to let F_x denote the event that x is the first such contact. The events F_x are disjoint, depend only on sites outside S_j, and satisfy

$$\sum_x \mathbb{P}(F_x) = \mathbb{P}(\mathcal{G}) > 1 - 3\delta. \tag{56}$$

Fix a potential contact x. Condition (48) applied at v_x supplies $\ell \geq R$ and $\gamma \in \mathscr{H}$ for which (48) holds. Consider the intersection of these three events:

(G1) local uniqueness between $v_x + \Lambda_m$ and $v_x + \Lambda_M$;

(G2) $v_x + \Lambda_m \overset{v_x+\Lambda_M}{\leftrightarrow} U_x$;

(G3) $v_x + \Lambda_m \overset{Q_\gamma(\ell,v_x)}{\leftrightarrow} F_\gamma(\ell, v_x)$.

Each failure has probability below δ^2. Hence their intersection, denoted G_x, has probability greater than $1-3\delta^2$.

Condition on the site states in S_j. If $\mathbb{P}(G_x \mid \omega|_{S_j}) > 1-\delta$, choose one realization of G_x with that shell state. Its path in (G2) ends at some $u \in U_x$. This path and the state of u are fixed by the shell state. In every other realization of G_x with the same shell state, choose a self-avoiding path in (G3) and follow its initial segment up to its first exit from $v_x + \Lambda_M$. The component of this segment inside the cube meets both $v_x + \Lambda_m$ and the inner boundary of $v_x + \Lambda_M$. The fixed path from (G2) has the same two properties, so (G1) joins the two paths inside the cube. It follows that

$$\mathbb{P}(u \overset{D}{\leftrightarrow} T \mid \omega|_{S_j}) > 1 - \delta. \tag{57}$$

Markov's inequality applied to $1 - \mathbb{P}(G_x \mid \omega|_{S_j})$ shows that shell states for which this conclusion fails have probability below 3δ. Let H_x be the event that an open $u \in U_x$ satisfying (57) exists. We have

$$\mathbb{P}(H_x) \geq 1 - 3\delta. \tag{58}$$

It remains to average without choosing an intermediate vertex in advance. For a shell pattern ξ, let

$$A_\xi = \left\{ u \in S_j : \xi(u) = 1,\ \mathbb{P}^\xi(u \overset{D}{\leftrightarrow} T) > 1 - \delta \right\}, \qquad \phi_\xi = \mathbb{P}^\xi(o \leftrightarrow A_\xi),$$

where $\mathbb{P}^\xi$ is the product site law with the shell fixed to ξ. Let $p_\xi = \mathbb{P}(\omega|_{S_j} = \xi)$. Lemma 8.2 shows that $F_x \cap H_x$ implies $o \leftrightarrow A_\xi$. Since F_x is determined outside S_j and H_x is determined by S_j, independence, (56), and (58) give the exact averaged estimate

$$\sum_\xi p_\xi \phi_\xi \geq \sum_x \mathbb{P}(F_x \cap H_x) = \sum_x \mathbb{P}(F_x)\mathbb{P}(H_x) > (1 - 3\delta)^2. \tag{59}$$

Call a pattern bad if $\phi_\xi \leq 1 - \delta_c$. Since $1 - (1 - 3\delta)^2 \leq 6\delta$, equation (59) gives

$$\delta_c \sum_{\xi \text{ bad}} p_\xi \leq \sum_\xi p_\xi (1 - \phi_\xi) < 6\delta \leq \frac{\varepsilon \delta_c}{2}.$$

The bad patterns therefore have total mass below $\varepsilon/2$.

For a good pattern, wire T to a forced-open target b. Every member of A_ξ is fixed open, and o is fixed open. Moreover, $\delta \leq \delta_c$ gives

$$\mathbb{P}^\xi(o \leftrightarrow A_\xi) > 1 - \delta_c, \qquad \min_{u \in A_\xi} \mathbb{P}^\xi(u \leftrightarrow b) > 1 - \delta_c.$$

Corollary 6.1 and (51) imply $\mathbb{P}^\xi(o \leftrightarrow T) > 1 - \varepsilon/2$. Averaging over good patterns gives

$$\mathbb{P}(o \leftrightarrow T) > (1 - \varepsilon/2)^2 > 1 - \varepsilon,$$

which proves (49). □

8.3 Elongated boxes

For a coordinate i, a sign $\sigma \in \{-1, 1\}$, and an integer $K \geq 2$, let $\gamma_{i,\sigma,K}$ be the geometry

$$\begin{aligned} Q &= \{x : -1 \leq \sigma x_i \leq K, |x_j| \leq 1\ (j \neq i)\}, \\ F &= \{x : \sigma x_i = K, |x_j| \leq 1\ (j \neq i)\}. \end{aligned}$$

Lemma 8.4 (Connection estimates for elongated boxes)**.** *Let $d \geq 2$ be an integer, and let $p \in (0,1)$ satisfy $\theta_{\text{site}}(p) > 0$. For every coordinate $1 \leq i \leq d$, every sign $\sigma \in \{-1, 1\}$, and every integer $K \geq 2$, the geometry $\gamma_{i,\sigma,K}$ satisfies the uniform connection property at p.*

Proof. We include the face chain because this is the first use of the uniformity of δ in Lemma 8.3. Fix a desired error α, and let $N = 8K - 4$. Put $\eta_N = \alpha$. Recursively, for $0 \leq t < N$, choose η_t to be the input tolerance supplied by the finite-volume connection-extension lemma with output tolerance η_{t+1}. Apply the lemma to the finite subface list at each of these N tolerances. Let $R_\star$ be the maximum of the resulting inflation radii, and put $\beta = \eta_0$. Condition (48) for radius $R_\star$ implies the same condition for each smaller radius, so these choices permit all N applications below. Choose k, n_1 so that a one of the selected subfaces of the boundary of Λ_L is reached from Λ_k with probability greater than $1 - \beta$ whenever $L \geq n_1$; this is Lemma 8.1(L2).

For a scale r, put

$$s = \lfloor r/8 \rfloor, \qquad L_0 = Kr - Ns, \qquad L_t = L_0 + ts, \qquad w_t = L_0 + tR_\star.$$

Take r so large that

$$s \geq 3K + 3KR_\star + k + n_1 + 8. \tag{60}$$

For $0 \leq t \leq N$, let

$$B_t = \{x : \sigma x_i = L_t, |x_j| \leq w_t \ (j \neq i)\}.$$

The initial subface estimate gives a connection from Λ_k to B_0. For $t < N$, use the finite site block

$$D_t = \{x : L_t - 2s \leq \sigma x_i \leq L_t + s, |x_j| \leq r \ (j \neq i)\}.$$

The wired source Λ_k is outside D_t. For $v \in B_t^{+R_\star}$ take

$$\ell(v) = L_t + s - \sigma v_i$$

and choose the transverse signs of the subface to point toward the coordinate axes. The inequalities in (60) give

$$\ell(v) \geq R_\star, \quad Q_\gamma(\ell(v), v) \subseteq D_t, \quad F_\gamma(\ell(v), v) \subseteq B_{t+1}.$$

Here are the bounds that verify the inclusions. Since $|\sigma v_i - L_t| \leq R_\star$, we have

$$s - R_\star \leq \ell(v) \leq s + R_\star.$$

The choice of s gives $s \geq 2R_\star$. Hence the longitudinal interval of $Q_\gamma(\ell(v), v)$ lies between $L_t - 2s$ and $L_t + s$. Transversely, (60) gives

$$w_N + s + 2R_\star \leq r.$$

Thus the full transverse cube lies in D_t. On the far face, choose each transverse orthant to point toward the corresponding coordinate hyperplane. Its coordinates have absolute value at most $\max\{w_t + R_\star, s + R_\star\} \leq w_{t+1}$, proving the final inclusion. Thus B_{t+1} satisfies (48) for $(B_t, D_t, R_\star, \mathscr{Q}_d)$. Induction on t gives

$$\mathbb{P}_p \left(\Lambda_k \overset{\Lambda_{L_0} \cup D_0 \cup \cdots \cup D_{t-1}}{\longleftrightarrow} B_t \right) > 1 - \eta_t,$$

with the union interpreted as Λ_{L_0} when $t = 0$. The connection-extension lemma is applied in the graph induced by the path region in this display together with D_t. The wired source remains outside D_t, and the new lattice subbox is D_t. The lemma therefore advances the displayed inequality from t to $t+1$ without allowing a path outside the next path region. Consequently B_N is reached with probability greater than $1 - \alpha$. The identities $L_N = Kr$ and $w_N \leq r$, which follow from (60), place B_N in the far face of the long box. All regions D_t lie in that box. Enlarging the source from Λ_k to Λ_m, $m \geq k$, proves (47) for $\gamma_{i,\sigma,K}$. □

8.4 Connections through adjacent blocks

The next lemma is the finite move used by the planar exploration. Its sets are recorded explicitly. For a center c, direction (i, σ), and scale r, put $c' = c + 20r\sigma e_i$ and

$$\begin{aligned}
D &= c + \{x : -5r \leq \sigma x_i \leq 25r, |x_j| \leq 5r \ (j \neq i)\}, \\
E &= c + \{x : 5r + 1 \leq \sigma x_i \leq 25r, |x_j| \leq 5r \ (j \neq i)\}, \\
H &= c + \{x : 5r \leq \sigma x_i \leq 22r, |x_j| \leq 2r \ (j \neq i)\}.
\end{aligned}$$

Lemma 8.5 (One-step connection across adjacent blocks)**.** *Let $d \geq 2$ be an integer, and let $p \in (0,1)$ satisfy $\theta_{\mathrm{site}}(p) > 0$. For each $\alpha > 0$, there are $\beta > 0$ and an integer $r_0 \geq 1$ such that, for every integer $r \geq r_0$, the following holds. Let G be a finite site model for which D is a finite site block at parameter p. Let $o \notin D$ be forced open, and put*

$$A = V(G) \setminus E, \qquad U = A \cup H.$$

Then

$$\mathbb{P}(o \overset{A}{\leftrightarrow} (c + \Lambda_{3r})) > 1 - \beta \quad \Longrightarrow \quad \mathbb{P}(o \overset{U}{\leftrightarrow} (c' + \Lambda_{3r})) > 1 - \alpha. \tag{61}$$

Proof. Choose all error tolerances before choosing the scale. Apply Lemma 8.3 to the aspect-88 long-box list with output error α, and denote its input tolerance by η_d. For $0 \leq j < d$, working backwards, let η_j be the input tolerance for the subface list with output error η_{j+1}. Let R be the maximum of the $d+1$ resulting inflation radii, and put $\beta = \eta_0$.

We first shrink the cross-section near c. Put $h = r + \lceil r/2 \rceil$. In a fixed ordering of the d coordinates, let B_j be the box centered at c whose first j half-widths are $h + jR$ and whose remaining half-widths are $3r + jR$. Assume that $r > 100(d+1)(R+1)$. We verify (48) with $T = B_{j+1}$, $B = B_j$, and $D = c + \Lambda_{5r}$. Let q be the coordinate whose width is reduced at this step, and write $h_1 = h + (j+1)R$. For $v \in B_j^{+R}$, choose the face in coordinate q that points toward c and use the scale

$$\ell(v) = R + \max\{0, |v_q - c_q| - h_1\}.$$

Its q-coordinate lies in $[-h_1, h_1] + c_q$. In each transverse coordinate choose the orthant that points toward the corresponding coordinate of c. Every transverse coordinate on the face then has absolute displacement at most $\max\{|v_k - c_k|, \ell(v)\}$, which is within the corresponding half-width of B_{j+1}. Moreover,

$$R \leq \ell(v) \leq 3r - h + R < 2r.$$

The bounds on v, together with $r > 100(d+1)(R+1)$, place the entire cube of radius $\ell(v)$ in $c + \Lambda_{5r}$. Thus the required covering condition holds from B_j to B_{j+1}. After d applications of the connection-extension lemma in the graph induced by A, the source reaches B_d, whose half-widths are smaller than $2r$. This restriction is valid because $c + \Lambda_{5r} \subseteq A$.

For the final move use the finite list of long-box geometries from Lemma 8.4 with aspect ratio 88. Work in

$$D' = c + \{x : -2r \leq \sigma x_i \leq 22r, |x_j| \leq 2r \ (j \neq i)\}.$$

For $v \in B_d^{+R}$ choose

$$\ell(v) = \left\lfloor \frac{20r - \sigma(v_i - c_i)}{88} \right\rfloor. \tag{62}$$

When $r \geq r_0$, the long box at this scale lies in D', and its far face lies in $c' + \Lambda_{2r}$. The lower bound containing the factor 100 absorbs the remainder in division by 88. Thus $c' + \Lambda_{2r}$ satisfies (48) for $(B_d, D', R, \mathscr{H}_{88})$. Pass now to the graph induced by $U = A \cup H$. The set D' is a finite site block in this graph, and the connection to B_d obtained in A remains valid. One more application of Lemma 8.3 reaches $c' + \Lambda_{2r}$. Enlarging the target to $c' + \Lambda_{3r}$ gives (61).

The induced graphs show that the first d steps use A and the last step adds only H. The backward choice of the $d+1$ tolerances gives the asserted number β. $\square$

8.5 Two-dimensional block layout

We give the final finite-to-infinite step in a form that exposes the measurability needed for site variables. Embed $\mathbb{Z}^2$ in two coordinates of $\mathbb{Z}^d$ and, for $z \in \mathbb{Z}^2$, set

$$c_z = 20rz, \qquad Q_z = c_z + \Lambda_{5r}, \qquad M_z = c_z + \Lambda_{3r}.$$

For an oriented nearest-neighbor edge $z \to w$ of the coarse lattice, let i and σ be its coordinate and sign. Its open strip between the two endpoint cubes is

$$B_{z,w} = c_z + \{x : 5r + 1 \le \sigma x_i \le 15r - 1, |x_k| \le 5r \ (k \ne i)\}.$$

The region exposed when w is examined from z is

$$E_{z,w} = B_{z,w} \cup Q_w = c_z + \{x : 5r + 1 \le \sigma x_i \le 25r, |x_k| \le 5r \ (k \ne i)\}. \tag{63}$$

The narrow connecting region is

$$H_{z,w} = c_z + \{x : 5r \le \sigma x_i \le 22r, |x_k| \le 2r \ (k \ne i)\}.$$

Choose $K \ge 20$, write $r = Ks$, and define the nested initial segments of this region

$$H^j_{z,w} = c_z + \{x : 5r \le \sigma x_i \le 5r + 10sj, |x_k| \le 2r \ (k \ne i)\}, \quad 0 \le j \le K. \tag{64}$$

For $1 \le j \le K$, their outer faces are

$$F^j_{z,w} = c_z + \{x : \sigma x_i = 5r + 10sj, |x_k| \le 2r \ (k \ne i)\}. \tag{65}$$

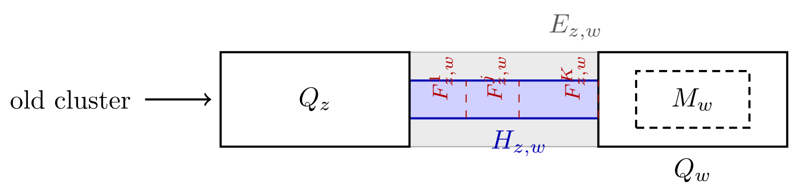

Figure 3: The regions used to examine the coarse-lattice edge $z \to w$, shown in the longitudinal coordinate. The broad edge region $E_{z,w}$ contains the endpoint cube Q_w. The narrower connecting region $H_{z,w}$ is revealed through nested faces. The examination stops at its first good face, so the unused part of the connecting region remains unqueried.

We now record the cover that controls every possible history. In the two planar coordinates, let

$$\mathrm{Cell}(z) = c_z + [-10r, 10r]^2 \times [-5r, 5r]^{d-2}.$$

For the directed edge $z \to w$, define the zone adjacent to the initial segment of its connecting region by

$$Z_{z,w} = c_z + \{x : 10r \le \sigma x_i \le 15r - 10s, |x_k| \le 2r \ (k \ne i)\}.$$

For a set $\mathcal{D}$ of previously examined coarse-lattice vertices, write

$$\mathrm{Cover}(\mathcal{D}) = \bigcup_{z \in \mathcal{D}} \left(\mathrm{Cell}(z) \cup \bigcup_{w:\, w \sim z} Z_{z,w} \right). \tag{66}$$

The next deterministic lemma verifies which sites remain unqueried. Two lattice sets are called separated when they are disjoint and no nearest-neighbor edge has one endpoint in each set.

Lemma 8.6 (Separation of block regions)**.** *Let $d \geq 2$, $K \geq 20$, and $s \geq 1$ be integers. Let $r = Ks$, and let $\mathcal{D} \subseteq \mathbb{Z}^2$ be a set of coarse-lattice vertices.*

(C1) *If $z \notin \mathcal{D}$, then* $\mathrm{Cover}(\mathcal{D})$ *and Q_z are separated.*

(C2) *If $z, w \notin \mathcal{D}$ and $z \sim w$, then* $\mathrm{Cover}(\mathcal{D})$ *and $E_{z,w}$ are separated.*

(C3) *If $w \sim z$, then $E_{w,z} \subseteq \mathrm{Cell}(w) \cup \mathrm{Cell}(z)$. For $0 \leq j < K$ and $y \sim z$,*

$$H^j_{z,y} \subseteq \mathrm{Cell}(z) \cup Z_{z,y}.$$

Proof. We compare the two planar coordinates. Distinct coarse-lattice centers differ by a multiple of $20r$ in at least one of them. A cell extends by $10r$ in each planar coordinate, whereas a cube extends by $5r$. Such a zone directed toward a neighboring center stops at level $15r - 10s$, while that neighbor's cube starts at level $15r$. These gaps are at least two lattice steps because $s \geq 1$. This proves (C1).

For (C2), first consider a cell or zone based at $u \in \mathcal{D}$. Since neither endpoint of the coarse-lattice edge zw belongs to $\mathcal{D}$, the vertex u differs from both endpoints. If the separating coarse-lattice coordinate is the direction of zw, then the preceding longitudinal bounds leave a gap of at least $5r$. Otherwise the transverse planar coordinate differs at both endpoints; the half-widths $10r$, $5r$, and $2r$ leave a gap of at least two lattice steps. The same comparison applies to a zone parallel to zw; for a perpendicular zone, use the two distinct planar coordinates. This proves separation from every set in the union (66).

Finally, the strip $B_{w,z}$ is covered by the two planar cells on its ends, and $Q_z \subseteq \mathrm{Cell}(z)$. This proves the first inclusion in (C3). If $j < K$, then

$$5r + 10sj \leq 15r - 10s.$$

The part of the initial segment below level $10r$ lies in $\mathrm{Cell}(z)$, and its remaining part lies in $Z_{z,y}$. This proves the second inclusion. $\square$

8.6 Adaptive block exploration

The preceding geometric construction supplies more than separation from the already inspected set. The next deterministic consequence is needed because a conditional connection estimate can otherwise become stale while a frontier vertex waits to be examined.

Definition 8.7 (Exploration state)**.** For the argument below, an exploration state is defined to consist of the following finite data.

(R1) A finite inspected set $\mathcal{E}$, together with its observed open subset $\xi \subseteq \mathcal{E}$.

(R2) Disjoint finite sets $\mathcal{O}, \mathcal{B}, \mathcal{R}$ of retained, rejected, and pending coarse-lattice vertices. Put $\mathcal{D} = \mathcal{O} \cup \mathcal{B}$.

(R3) For each $z \in \mathcal{O}$, an ξ-open nearest-neighbor path $\pi_z \subseteq \mathcal{E}$ from 0 to a vertex $a_z \in M_z$.

(R4) For each $z \in \mathcal{R}$, a unique parent $\mathrm{par}(z) \in \mathcal{O}$ and a time $t(z)$ at which z was entered the pending set. At that time a finite event $\mathscr{I}_z$, depending only on the then inspected coordinates and on the still unread coordinates of $E_{\mathrm{par}(z),z}$, was recorded, with

$$\mathscr{I}_z = \left\{ 0 \overset{\mathcal{E}_{t(z)} \cup E_{\mathrm{par}(z),z}}{\leftrightarrow} M_z \right\}, \qquad \mathbb{P}(\mathscr{I}_z \mid \mathcal{F}_{t(z)}) > 1 - \delta_c. \tag{67}$$

A coarse-lattice vertex is called *untested* when it belongs to $\mathcal{R}$ or to none of $\mathcal{O}, \mathcal{B}, \mathcal{R}$. A pending vertex is examined only through its recorded parent edge. Until that examination, no other operation is allowed to inspect an unread site of its assigned region $E_{\mathrm{par}(z),z}$.

Lemma 8.8 (Independence of unqueried regions)**.** *The blocks, edge regions, and initial segments of the connecting regions constructed above satisfy the following exploration rule. Suppose that every new vertex enters the pending set when it is first offered by a retained neighbor, and that all later examinations ignore edges whose other endpoint is already pending. Then the following statements hold.*

(Q1) *Distinct pending vertices have disjoint assigned unread regions.*

(Q2) *An examination not directed at a pending vertex z reads no unread site of $E_{\mathrm{par}(z),z}$.*

(Q3) *Consequently, immediately before z is examined,*

$$\mathbb{P}\left(0 \overset{\mathcal{E}\cup E_{\mathrm{par}(z),z}}{\leftrightarrow} M_z \middle| \mathcal{F}\right) > 1 - \delta_c. \tag{68}$$

(Q4) *Every site read while examining z lies in* $\mathrm{Cover}(\mathcal{D} \cup \{z\})$*, and each site read for the first time has, conditional on the complete past, its original independent Bernoulli law with parameter p.*

Proof. The assertion is deterministic once the exploration state is fixed. The interior of an edge region belongs to the tube of its unoriented coarse-lattice edge. Tubes of nonincident coarse-lattice edges are disjoint. Tubes of incident coarse-lattice edges can meet only in the endpoint cell and in the initial segments associated with their directions. If an endpoint is pending, all nonparent edges incident to that endpoint are ignored until that vertex is examined. Hence the unread part of its parent tube is not inspected by another examination. These observations, together with Lemma 8.6(C1)–(C3), prove (Q1), (Q2), and (Q4). This is a finite coordinate check: after translation and the dihedral symmetries of the first two coordinates, the only cases are equal edges, edges with one common endpoint, parallel disjoint edges, and perpendicular disjoint edges. The defining coordinate intervals are respectively nested only in the first case, meet only in the declared endpoint cell in the second, and are separated by at least one lattice layer in the last two cases.

For (Q3), let $\mathcal{F}_0$ be the history at which z entered the pending set and let $\mathcal{G}$ be the information subsequently read before its examination. By (Q2), $\mathcal{G}$ is generated by coordinates disjoint from the unread coordinates on which $\mathscr{I}_z$ depends. Product independence gives

$$\mathbb{P}(\mathscr{I}_z \mid \mathcal{F}_0 \vee \mathcal{G}) = \mathbb{P}(\mathscr{I}_z \mid \mathcal{F}_0) > 1 - \delta_c.$$

The presently allowed region $\mathcal{E} \cup E_{\mathrm{par}(z),z}$ contains the region used in $\mathscr{I}_z$. Connectivity is increasing under enlargement of the allowed region, so (68) follows. □

Lemma 8.9 (Conditional success of one exploration step)**.** *Let $d \geq 2$ and $p \in (0,1)$ satisfy $\theta_{\mathrm{site}}(p) > 0$. For every $0 < \rho \leq 2^{-32}$ there are integers $K \geq 20$ and $s \geq 1$, with $r = Ks$, and an examination rule with the following property. Conditional on all sites of Q_0 being open, every exploration state from the preceding definition can be extended by examining its first pending vertex z, with parent w, so that:*

(P1) *Conditional on the complete pre-examination history, the probability that the examination is successful is greater than $1 - \rho/2$.*

(P2) *On the successful event, the newly inspected configuration contains an actual open path* π_z *from* 0 *to some* $a_z \in M_z$*; in particular* $\|a_z - c_z\|_\infty \le 3r$.

(P3) *For every neighbor* y *of* z *that was neither tested nor pending at the start of the examination, the successful outcome supplies the conditional estimate*

$$\mathbb{P}\left(0 \overset{\mathcal{E}' \cup E_{z,y}}{\leftrightarrow} M_y \middle| \mathcal{F}'\right) > 1 - \delta_c, \tag{69}$$

where $\mathcal{F}'$ *is the complete post-examination history and* $\mathcal{E}'$ *its inspected set.*

(P4) *The conclusions of Lemma 8.8 continue to hold after the examination, whether or not it is successful.*

(P5) *Once* p*, the geometry, and the pre-examination history are fixed, the examination is a deterministic adaptive decision tree that reads at most*

$$M_\star = |E_{0,e_1}| + 4|H_{0,e_1}|$$

previously unread site coordinates.

The initial state has 0 *retained,* $\pi_0 = (0)$*, and the four neighbors of* 0 *pending with parent* 0 *and with conditional estimates of the form* (67).

Proof. We choose the error budget with room for the realized incoming connection. Apply Lemma 8.5 with output error $\rho/32$, and let δ_1 be its input tolerance. Put

$$\delta_c = \min\{\delta_1, \rho/4, 1/2\}.$$

Apply Lemma 8.3 with output error δ_c, and let $\delta_2 > 0$ be its input tolerance. Increase K, if necessary, so that

$$(1 - \delta_2)^K + \frac{\rho}{32} \le \frac{\rho}{16}. \tag{70}$$

Choose s and then $r = Ks$ exactly as in the construction above: all aspect-$2K$ elongated-box geometries used below satisfy the hypotheses of Lemma 8.3, all aspect-6 initial geometries satisfy the uniform connection property with error δ_c, and $2R \le s$ for the relevant inflation radius R.

For the initial history, long-box hitting gives, for every neighbor z of 0,

$$\mathbb{P}\left(0 \overset{Q_0 \cup E_{0,z}}{\leftrightarrow} M_z \middle| Q_0 \text{ is open}\right) > 1 - \delta_c.$$

The four assigned regions are disjoint outside Q_0, so the neighbors may enter the pending set simultaneously.

Now consider a pending vertex z with parent w. By Lemma 8.8, the incoming event

$$A_z = \left\{0 \overset{\mathcal{E} \cup E_{w,z}}{\leftrightarrow} M_z\right\}$$

has conditional probability greater than $1 - \delta_c$. Reveal every previously unread site of $E_{w,z}$. The event A_z is now determined. On A_z, select the lexicographically first self-avoiding open path in the finite induced graph $\mathcal{E} \cup E_{w,z}$ from 0 to M_z, and call it π_z; let a_z be its terminal vertex. This is a path in the actually observed configuration, not in a positive-probability completion.

Let X be the set of neighbors y of z that are neither tested nor pending. For each $y \in X$, reveal the nested initial segment of the connecting region one layer at a time. At level j, after pinning all states read so far, call the level good when

$$\mathbb{P}\left(0 \overset{\mathcal{E}\cup E_{w,z}\cup H^j_{z,y}\cup E_{z,y}}{\leftrightarrow} M_y \middle| \text{current history}\right) > 1-\delta_c. \tag{71}$$

Stop at the first good level, or after level $K-1$ if no good level exists. The examination is declared successful precisely when A_z occurs and a good level is found for every $y \in X$.

Fix $y \in X$. Before the examination, restrict the pinned model to the graph induced by $\mathcal{E} \cup E_{w,z} \cup E_{z,y}$. The assigned region of y has not been read, and Lemma 8.6 makes $Q_z \cup E_{z,y}$ the finite site block D of Lemma 8.5; the previously available region is its source side A. The incoming estimate and Lemma 8.5 give

$$\mathbb{P}\left(0 \overset{\mathcal{E}\cup E_{w,z}\cup H_{z,y}}{\leftrightarrow} M_y \middle| \text{past}\right) > 1-\frac{\rho}{32}. \tag{72}$$

For $0 \le j < K$, let B_j be the event that level j is not good and put

$$A_j = \left\{0 \overset{\mathcal{E}\cup E_{w,z}\cup H^{j+1}_{z,y}}{\leftrightarrow} F^{j+1}_{z,y}\right\}.$$

We claim that on B_j, conditional on the history through the initial segment of depth j,

$$\mathbb{P}(A_j \mid \text{history through depth } j) \le 1-\delta_2. \tag{73}$$

To verify the claim, use the still uninspected box

$$D_j = c_z + \{x : 5r + 10sj + 1 \le \sigma x_i \le 25r,\ |x_k| \le 5r\ (k \ne i)\}.$$

The block-separation and unqueried-region lemmas imply that every site of D_j not already fixed on its lower boundary is a fresh parameter-p site. After pinning the history through $H^j_{z,y}$, restrict to the pinned source region together with D_j. This is a finite site block, and its lower boundary is adjacent to the terminal face of $H^j_{z,y}$.

We next verify (48) with $T = M_y$, $B = F^{j+1}_{z,y}$, and $D = D_j$. Use the aspect-$2K$ elongated-box geometry and put

$$a_j = 5r + 10s(j+1), \qquad \ell_j = \left\lfloor \frac{20r - a_j}{2K} \right\rfloor.$$

For v in the R-enlargement of $F^{j+1}_{z,y}$,

$$a_j - R \le \sigma(v-c_z)_i \le a_j + R, \qquad |(v-c_z)_k| \le 2r + R \quad (k \ne i).$$

The relations $0 \le j < K$, $r = Ks$, $K \ge 20$, and $2R \le s$ give

$$R \le \ell_j \le \frac{15s}{2}, \qquad \ell_j + R \le 8s < 10s.$$

Thus the long box based at v begins strictly after the lower face of D_j, its forward coordinate is at most $20r + R < 25r$, and every transverse coordinate has absolute displacement at most $2r + R + \ell_j \le 3r$. If $q_j = 2K\ell_j$, then the defining property of the floor gives

$$20r - R - (2K-1) \le \sigma(v-c_z)_i + q_j \le 20r + R.$$

This interval is contained in $[17r, 23r]$, so the far face lies in M_y; the preceding bounds put the whole elongated box in D_j. Condition (48) holds uniformly over $(F_{z,y}^{j+1})^{+R}$.

If the conditional probability of A_j were greater than $1-\delta_2$, Lemma 8.3 in this finite block would make the level good, contradicting B_j. This proves (73). Iterated conditioning gives

$$\mathbb{P}\left(\bigcap_{j=0}^{K-1}(A_j \cap B_j)\middle|\text{past}\right) \leq (1-\delta_2)^K. \tag{74}$$

No old inspected site is adjacent to $E_{z,y}$, and $E_{w,z}$ reaches only the source side of $F_{z,y}^1$. Every path in the event of (72) therefore meets $F_{z,y}^1, \dots, F_{z,y}^K$ in order. Its prefix to $F_{z,y}^{j+1}$ is contained in $\mathcal{E} \cup E_{w,z} \cup H_{z,y}^{j+1}$. Hence, if every level is bad and the event in (72) occurs, every A_j occurs. Combining (72) and (74) yields

$$\mathbb{P}(\text{no good level for } y \mid \text{past}) \leq \frac{\rho}{32} + (1-\delta_2)^K \leq \frac{\rho}{16}.$$

There are at most four possible directions. Thus

$$\begin{aligned}\mathbb{P}(\text{examination unsuccessful} \mid \text{past}) &\leq \mathbb{P}(A_z^c \mid \text{past}) + \sum_{y\in X}\mathbb{P}(\text{no good level for } y \mid \text{past})\\ &\leq \frac{\rho}{4} + 4\frac{\rho}{16} = \frac{\rho}{2} < \rho.\end{aligned}$$

The displayed bound is at most $\rho/2$, so the success probability is greater than $1-\rho/2$. This proves (P1), and the construction of π_z proves (P2).

On the successful event, record for y the first good level found above. The post-examination conditional probability in (71) is exactly the estimate required to place y in the pending set with parent z. Conditioning on initial segments belonging to other directions does not alter it, since the corresponding unread coordinate sets are disjoint; the final allowed region only grows. This proves (P3). The deterministic cover and freshness assertions, including histories on which A_z or a good-level test fails, follow from Lemma 8.8. Thus (P4) holds. During one examination we reveal only the unread part of the incoming edge region and, in at most four directions, subsets of the corresponding full narrow connecting regions. Translation invariance gives the bound in (P5). □

Lemma 8.10 (Adaptive Bernoulli thinning)**.** *Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. At step n, let S_n be a $\{0,1\}$-valued outcome, determined after a finite examination, and let*

$$q_n = \mathbb{P}(S_n = 1 \mid \mathcal{F}_{n-1}) \geq q_0 > 0.$$

Let U_n be independent uniform random variables, independent of all site coordinates and of the preceding uniforms, and put

$$R_n = \mathbf{1}\{S_n = 1\}\mathbf{1}\{U_n \leq q_0/q_n\}.$$

Then, in the adaptive order in which the steps are requested, the variables R_n are independent Bernoulli variables of parameter q_0.

Proof. For every event $H \in \mathcal{F}_{n-1}$,

$$\mathbb{P}(H, R_n = 1) = \mathbb{E}\left[\mathbf{1}_H q_n \frac{q_0}{q_n}\right] = q_0\mathbb{P}(H).$$

The same identity with $R_n = 0$ follows by subtraction. Induction on n therefore factors every finite joint atom into the Bernoulli-q_0 product. The assertion is unaffected by an adaptive choice of the next item because that choice is $\mathcal{F}_{n-1}$-measurable. □

Lemma 8.11 (Comparison of finite decision trees at two parameters)**.** *Let $\mathcal{T}$ be a deterministic adaptive decision tree that queries at most M previously unread Bernoulli coordinates and then returns a Boolean output S. For $0 \le q \le p \le 1$,*

$$\mathbb{P}_q(S = 1) \ge \mathbb{P}_p(S = 1) - M(p - q). \tag{75}$$

The same estimate holds conditionally after any fixed finite transcript, provided the queried coordinates are fresh under both product laws.

Proof. Attach one independent uniform variable to every coordinate and declare it open at parameter t when the uniform is at most t. Run the p- and q-trees in lockstep until their first disagreement. Before a disagreement they have the same transcript and therefore query the same next coordinate. At each of at most M queries, the two answers differ only when the corresponding uniform lies in $(q, p]$, an event of probability $p - q$. A union bound gives probability at most $M(p - q)$ of any disagreement. On the complementary event the two outputs coincide, which proves (75). Conditioning on a fixed transcript only removes already fixed coordinates and leaves the same coupling on the fresh ones. □

Lemma 8.12 (Stability of a fixed exploration rule under parameter change)**.** *Fix $d \ge 2$, $p \in (0, 1)$ with $\theta_{\mathrm{site}}(p) > 0$, and $0 < \rho \le 2^{-32}$. Fix the scales and the examination rule supplied by Lemma 8.9. In particular, continue to call a level "good" according to the conditional probability computed at the reference parameter p. Run this same rule while the fresh physical sites have parameter $q \le p$.*

Every history produced by this fixed rule is an exploration state satisfying the conditional estimates computed at parameter p. If S_z is the success indicator for the next examination, then

$$\mathbb{P}_q(S_z = 1 \mid \textit{pre-examination history}) > 1 - \frac{\rho}{2} - M_\star(p - q), \tag{76}$$

where $M_\star$ is the bound in Lemma 8.9(P5). On $S_z = 1$, all path and pending-state conclusions of that lemma hold in the actual parameter-q configuration.

Proof. The initial four pending-state estimates follow from the parameter-p bounds used in Lemma 8.9. Inductively, a child enters the pending set only at a level that the fixed rule declares good, so its recorded inequality is again an estimate at parameter p. The argument for the assigned unqueried regions is deterministic; when interpreted under the reference parameter-p product law, observations made in disjoint regions do not alter the estimate. Hence every history generated at parameter q is also admissible for the rule and estimates selected at parameter p.

For a fixed such history, the examination is a decision tree on at most $M_\star$ fresh coordinates. Under parameter p its success probability is greater than $1 - \rho/2$ by Lemma 8.9(P1). Lemma 8.11 gives (76). Success is defined through actually observed open paths and through the same recorded unqueried regions, so its deterministic conclusions are unchanged when the sampling parameter is q. □

Proposition 8.13 (Percolation at a smaller parameter)**.** *Let $d \ge 2$ and $p \in (0, 1)$. If $\theta_{\mathrm{site}}(p) > 0$, then there exists $q \in (0, p)$ such that $\theta_{\mathrm{site}}(q) > 0$.*

Proof. Set

$$\rho = 2^{-33}, \qquad q_0 = 1 - \rho.$$

Then $0 < \rho \le 2^{-32}$ and, since $2^{33} > 1000$, $q_0 > 999/1000 = p_\star$. Apply Lemma 8.9 at the reference parameter p, and let $M_\star$ be its uniform read bound. Set

$$\varepsilon_p = \min\left\{\frac{p}{2}, \frac{\rho}{4M_\star}\right\}, \qquad q = p - \varepsilon_p.$$

Then $0 < q < p$ and $M_\star(p-q) \le \rho/4$. Run the fixed rule selected at parameter p with fresh sites sampled at parameter q. Lemma 8.12 gives, at every requested examination,

$$\mathbb{P}_q(S_z = 1 \mid \text{complete past}) > 1 - \frac{3\rho}{4} > q_0.$$

Condition on Q_0 being open. Retain the origin of the coarse lattice, put its four neighbors in the pending set, and maintain the deterministic queue from Section 8.6. For the next requested vertex z, put

$$r_z = \mathbb{P}_q(S_z = 1 \mid \text{complete pre-examination history}) > q_0.$$

Using a fresh independent uniform variable, retain z precisely when $S_z = 1$ and the uniform is at most q_0/r_z. Lemma 8.10 shows that the retained indicators, in adaptive query order, are independent Bernoulli-q_0 variables. The queue is therefore exactly the deterministic exploration of the origin cluster in site percolation on the coarse lattice $\mathbb{Z}^2$ at parameter q_0, conditional on its origin being open.

Fix n and stop when the queue empties or a retained coarse-lattice vertex reaches $\partial[-n,n]^2$. The construction is finite, so

$$\begin{aligned}&\mathbb{P}_q^{(d)}\left(\text{a retained vertex reaches } \partial[-n,n]^2 \middle| Q_0 \text{ is open}\right)\\&= \mathbb{P}_{q_0}^{(2)}\left(0 \leftrightarrow \partial[-n,n]^2 \mid 0 \text{ is open}\right).\end{aligned} \tag{77}$$

By Proposition 7.8,

$$\eta := \mathbb{P}_{q_0}^{(2)}(0 \leftrightarrow \infty \mid 0 \text{ is open}) > 0,$$

so the right side of (77) is at least η for every n.

On the retained event, Lemma 8.9(P2) supplies an actually observed parameter-q open path from 0 to a point $a_z \in M_z$ for some $z \in \partial[-n,n]^2$. Hence

$$\|a_z\|_\infty \ge 20rn - 3r.$$

By Lemma 8.8(Q4) and the definitions of $\mathrm{Cell}(z)$ and $Z_{z,w}$, this path lies in

$$\mathbb{S}_r = \mathbb{Z}^2 \times [-5r, 5r]^{d-2},$$

where the last factor is a singleton when $d = 2$. Therefore

$$\mathbb{P}_q^{(d)}\left(0 \overset{\mathbb{S}_r}{\leftrightarrow} \mathbb{Z}^d \setminus \Lambda_{20rn-3r} \middle| Q_0 \text{ is open}\right) \ge \eta$$

for every n. These events decrease to $\{0 \overset{\mathbb{S}_r}{\leftrightarrow} \infty\}$. Continuity from above yields

$$\mathbb{P}_q^{(d)}\left(0 \overset{\mathbb{S}_r}{\leftrightarrow} \infty \middle| Q_0 \text{ is open}\right) \ge \eta.$$

Since $\mathbb{P}_q^{(d)}(Q_0 \text{ is open}) = q^{|Q_0|} > 0$, the unrestricted parameter-q model percolates. Thus $\theta_{\mathrm{site}}(q) > 0$. □

Corollary 8.14 (Openness of the percolating phase)**.** *For every $d \ge 2$, the set*

$$\{p \in (0,1) : \theta_{\mathrm{site}}^{(d)}(p) > 0\}$$

is open in $(0,1)$.

Proof. If $\theta_{\mathrm{site}}^{(d)}(p) > 0$, Proposition 8.13 gives $q < p$ with $\theta_{\mathrm{site}}^{(d)}(q) > 0$. Monotonicity then gives positive percolation probability at every parameter in $(q,1)$, which contains a neighborhood of p in $(0,1)$. □

9 Criticality

Proof of Theorem 1.1. Fix $d \geq 2$ and put

$$p = p_c^{\mathrm{site}}(\mathbb{Z}^d).$$

Lemma 7.9 gives $p \in (0, 1)$. If $\theta_{\mathrm{site}}(p) > 0$, Proposition 8.13 produces $q < p$ with $\theta_{\mathrm{site}}(q) > 0$. This contradicts

$$p = \inf\{t : \theta_{\mathrm{site}}(t) > 0\}.$$

Therefore

$$\theta_{\mathrm{site}}\left(p_c^{\mathrm{site}}(\mathbb{Z}^d)\right) = 0.$$

$\square$

10 Relation to the Lean development

The proof separates into finite and lattice parts. The following dependency map records a useful organization for comparing the mathematical argument with the accompanying Lean development. It is not a claim that each displayed derivation in the manuscript has a line-by-line Lean counterpart. The current Lean-checked final theorem has dimension hypothesis $d \geq 3$, whereas the mathematical statements in this manuscript are formulated for $d \geq 2$.

One useful finite dependency order is as follows.

1. Define finite labeled hypergraphs, induced deletion, hyperedge clusters, set clusters, and product weights.

2. Prove the exploration identity in Lemma 2.2 and exact conditioning on a cluster.

3. Prove the one-cluster and two-cluster inequalities, Lemmas 3.4 and 3.5.

4. Retain source labels together with their vertex incidence sets and prove Lemma 4.3 by strong induction on the number of vertices. The formal statement must quantify over the label type and incidence map before the induction begins.

5. Formalize the iterated conditioned-covariance estimate. Only its two-source averaging step should invoke the labeled lemma; the remaining finite-sum identities are algebraic.

6. Derive Theorems 5.1 and 5.4, then formalize the site-to-hyperedge representation and Corollary 6.1.

A useful regression test is a pair of distinct incident hyperedges with the same outside incidence set, as in Figure 2. A formal proof that pushes exposed states directly to vertex traces will incorrectly replace an inclusion by an equality. The source labels must remain distinct through the union and intersection in (14) and (15).

For the lattice part, first formalize the translation zero–one law, finite modification, trifurcation surgery, external-boundary connectivity, and the star-connected-set count. Then formulate analogous finite-volume connection events and block-chaining estimates for site configurations. Conditioning pins vertices rather than edges. In the estimate leading to (54), close the at most $2dN$ fresh interior neighbor sites. In the all-open-set estimate, use disjoint finite sets J_x. Formalize Lemma 8.6 before the planar exploration; its separation property keeps each outgoing finite site block fresh. Finally formalize the finite decision-tree coupling before comparing the same exploration rule at nearby parameters.

A Selected statements for formalization

This appendix records statements useful for comparing the mathematical argument with the proof-assistant development. It is explanatory: the prose and Lean sources do not correspond line by line, and the appendix does not extend the dimension range of the Lean theorem beyond $d \geq 3$.

A.1 Finite conditional laws

For a finite source S and a proposed support K, partition the hyperedge labels into those contained in K, those crossing from K to its complement, and those disjoint from K. On $\{C_S = K\}$ every crossing label is closed, and the disjoint labels retain their product weights. Thus, for every outside state η,

$$\mathbb{P}(C_S = K, \omega_{E_{\text{out}}(K)} = \eta) = \mathbb{P}(C_S = K) \prod_{e \in E_{\text{out}}(K)} p_e^{\eta_e}(1 - p_e)^{1-\eta_e}.$$

All later statements that an unexplored model is independent are to be expanded through this identity.

For disjoint supports K, L containing source sets S, T, respectively, cluster exposure commutes:

$$\mathbb{P}(C_S = K)\mathbb{P}_{V \setminus K}(C_T = L) = \mathbb{P}(C_T = L)\mathbb{P}_{V \setminus L}(C_S = K) = \mathbb{P}(C_S = K, C_T = L).$$

This equality for two disjoint clusters is used in the iterated conditional covariance inequality.

A.2 One-level conditioned covariance identity

Let S contain x, with $Y, S, \{z\}$ otherwise disjoint, and expose C_Y under $x \not\leftrightarrow Y$. If

$$\overline{h}_S(u) = \mathbb{E}\left[\mathrm{Cov}_{V \setminus C_Y}(g(C_x), \mathbf{1}\{u \leftrightarrow S\}) \mid x \not\leftrightarrow Y\right],$$

$D_z = \{z \not\leftrightarrow Y \cup S\}$, $c_z(u) = \mathbb{P}(u \in C_z \mid D_z)$, and Φ is the increasing residual functional in (25), then the exact, cleared form is

$$\begin{aligned} &\mathbb{P}(x \not\leftrightarrow Y)\big(\overline{h}_S(u) - c_z(u)\overline{h}_S(z) - \overline{h}_{S \cup \{z\}}(u)\big) \\ &\quad = \mathbb{P}(D_z)\,\mathrm{Cov}\big(\Phi(C_z), \mathbf{1}\{u \in C_z\} \mid D_z\big). \end{aligned}$$

To formalize it, expand both covariances, use the displayed disjoint decomposition of the connection event, and rewrite the double cluster sum with the equality for two disjoint clusters above. No measure-theoretic disintegration is required because every sample space is finite.

A.3 Exploration conventions

Every adaptive exploration history contains the complete list of queried sites or hyperedges and their states. A reached vertex set alone is not used as a sigma-field. Conditional inequalities are stated after multiplication by the conditioning probability. Named-vertex connectivity in an induced hypergraph is false when the named vertex is absent.

The estimate used to obtain many separated contacts is an application of the following stopping lemma. If, at the t-th failure, a predictably selected set of at most m fresh Bernoulli-p sites would block the terminal event when all closed, then

$$\mathbb{P}(A, X \geq t) \leq \big(1 - (1-p)^m\big)^t.$$

The proof is induction using conditional expectation at the failure stopping times.

The planar domination step uses adaptive thinning: if a queried success has conditional probability $q_n \geq q_0$, retain it using an independent uniform variable with probability q_0/q_n. The retained variables are independent Bernoulli-q_0 in adaptive query order.

A.4 Scope of the formalization

No percolation-specific critical theorem is imported. Translation ergodicity, finite-energy uniqueness, high-density planar percolation, and interiority of the critical parameter are proved in Section 7. The final contradiction uses the proposition establishing percolation at a smaller parameter, not a half-space theorem. A foundational formalization still requires the standard construction of countable product measure, approximation by cylinder events, conditional expectation on finite sigma-fields, and continuity from above. These sentences describe the dependency boundary of the mathematical manuscript; they do not assert that every argument above, or the $d = 2$ case, has been checked by Lean.

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