Tomography

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November 2, 2014

Tomography

- goal is to reconstruct or estimate a function $f : \mathbf{R}^2 \to \mathbf{R}$ from (possibly noisy) line integral measurements
- f is sometimes called (or interpreted as) an *image*
- we'll focus on 2-dimensional case, but it can be extended to 3-
- used in medicine, manufacturing, networking, geology
- best known application: CAT (computer-aided tomography) scan

Outline

Line integral measurements

Least-squares reconstruction

Example

Line integral

- parameterize line ℓ in $\mathbf{R}^2$ as

$$z(t) = (x_0, y_0) + t(\cos\theta, \sin\theta), \quad t \in \mathbf{R}$$

  - (x_0, y_0) is (any) point on the line
  - θ is angle of line (measured from horizontal)
  - parameter t is length along line
- line integral (of f, on ℓ) is

$$\int_\ell f = \int_{-\infty}^{\infty} f(z(t))\, dt$$

Line integral measurements

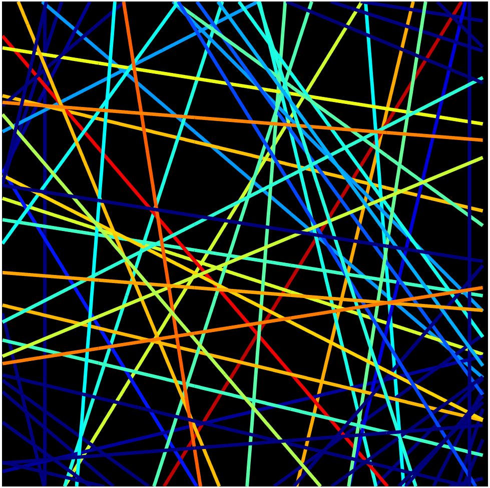
- we have m line integral measurements of f with lines $\ell_1, \ldots, \ell_m$
- ith measurement is

$$y_i = \int_{-\infty}^{\infty} f(z_i(t))\, dt + v_i, \quad i = 1, \ldots, m$$

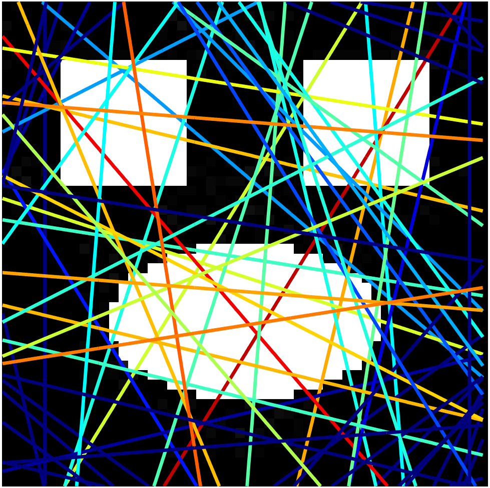
  - $z_i(t)$ is parametrization of ℓ_i
  - v_i is the *noise* or *measurement error* (assumed to be small)
- line integral measurements are $y = (y_1, \ldots, y_m)$

Example of line integral measurements in 2-d

image is 50×50
60 measurements shown

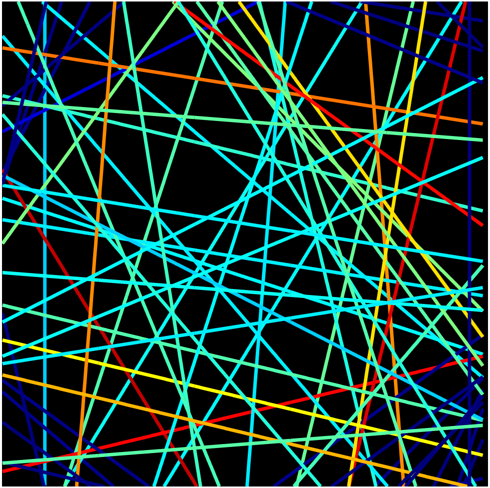


Example of line integral measurements in 2-d

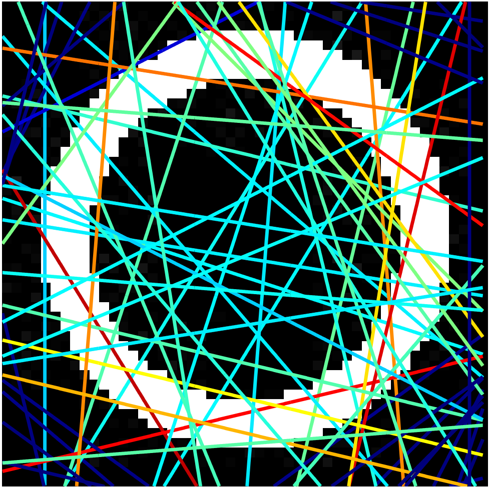


Another example

image is 50×50
60 measurements shown



Another example



Discretization of f

- we discretize the function f, assuming it's constant on n pixels, numbered 1 to n
- represent (discretized) function f by n-vector x
- x_i is value of f in pixel i
- line integral measurement y_i has form

$$y_i = \sum_{j=1}^{n} A_{ij} x_j + v_i$$

- A_{ij} is length of line ℓ_i in pixel j
- in matrix-vector form, we have

$$y = Ax + v$$

Outline

Smoothness prior

- we assume that image is not too rough, as measured by (Laplacian)

$$\|D_1x\|^2 + \|D_2x\|^2$$

  - D_1x gives first order difference in horizontal direction
  - D_2x gives first order difference in vertical direction

- roughness measure is sum of squares of first order differences

- it is zero only when x is constant

Least-squares reconstruction

- choose $\hat{x}$ to minimize

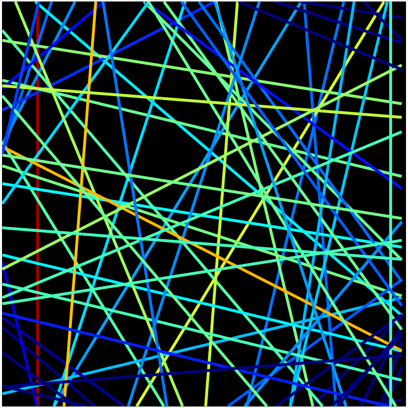
$$\|A\hat{x} - y\|^2 + \lambda(\|D_1\hat{x}\|^2 + \|D_2\hat{x}\|^2)$$

  - first term is $\|v\|^2$, or deviation between what we observed (y) and what we would have observed without noise ($A\hat{x}$)
  - second term is roughness measure
- regularization parameter $\lambda > 0$ trades off measurement fit versus roughness of recovered image

Outline

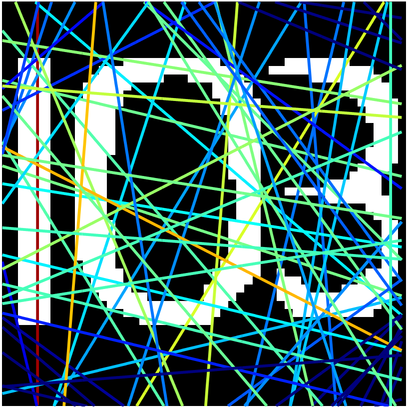
Image

50×50 image ($n = 2500$)



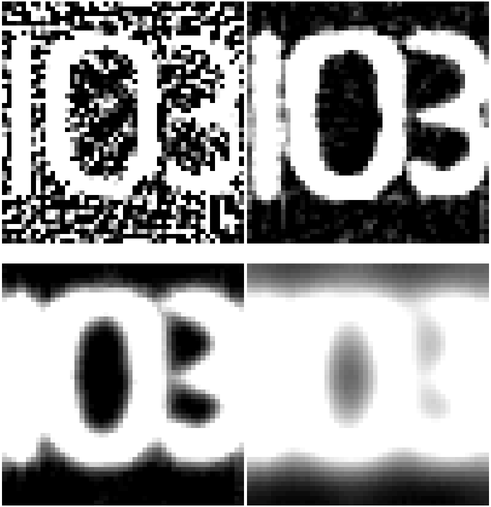
Lines

- 40 angles, 40 offsets ($m = 1600$ lines)
- 60 lines shown
- small measurement noise



Reconstruction

reconstructions with $\lambda = 10^{-6}, 20, 230, 2600$



Varying the number of line integrals

reconstruct with $m = 100, 400, 2500, 6400$ lines (with $\lambda = 10, 15, 25, 30$)